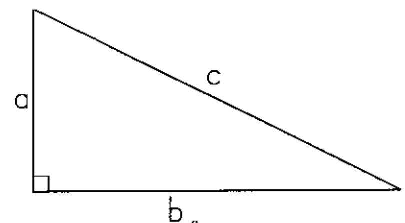
Review 10 — Trigonometry I - Mostly Right Triangles

Triangles occur not only in many problems of importance to physicists, but in those of the practical arts such as navigation and mechanical design. For example, a triangular structure formed by three rigid straight rods joined end-to-end has the special property that it cannot be squashed out of shape (as a rectangle can) without bending or breaking the rods. The vertex angles can only have certain specific values, depending on the lengths of the three sides of the triangle. Trigonometry deals with interrelationships among the sides and angles of triangles. In particular, right triangles merit special attention.

IMPORTANT SPECIAL TRIANGLES AND THEIR PROPERTIES

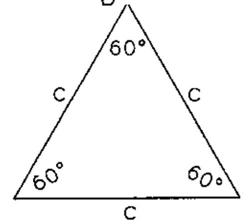
Since certain special shaped triangles occur repeatedly in physics problems their most important properties should be well understood.



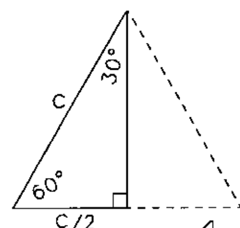
Right triangles—Pythagorean theorem. The lengths of the legs a and b are related to the length of the hypotenuse c according to

$$c^2 = a^2 + b^2 .$$

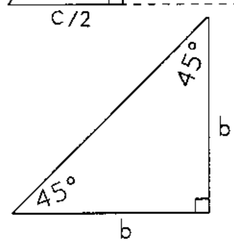
This gem-like rule was known by the Babylonians and proven to be exactly correct by the ancient Greeks. Yet it crops up even in the most advanced contexts in physics.



The equilateral triangle. This triangle is completely symmetric. All sides are equal in length. All angles are equal; each angle is 60°.



30–60–90 triangle. Split an equilateral triangle exactly in half. Each piece is a right triangle which is a favorite of writers of physics problems. The acute angles are 30° and 60°; the length of the side opposite the 30° angle is half that of the hypotenuse.



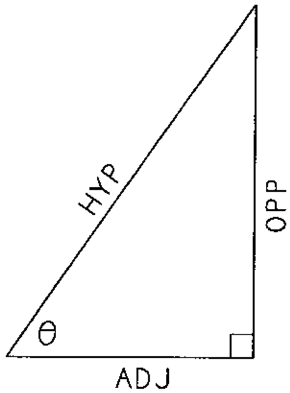
Right isosceles triangle. The acute interior angles are each 45°.

TRIGONOMETRIC FUNCTIONS

Apart from the useful facts about special triangles outlined above, trigonometry provides us with methods to relate sides and angles in triangles of any shape. However, instead of using values of angles themselves in its formulas, trigonometry relies on sets of numbers, called "trigonometric functions", which depend on values of angle.

Physics problems almost exclusively require only three trigonometric ("trig") functions in the solution of problems. These are the *sine*, the *cosine*, and the *tangent*. These functions can be defined in terms of the properties of a right triangle, as given in the next paragraph.

Sine, cosine, and tangent. Imagine acute angle θ at one vertex of a right triangle, as illustrated here. The hypotenuse and the legs of the triangle opposite and adjacent to θ have been labelled HYP, OPP, and ADJ respectively. The sine, cosine, and tangent of θ (abbreviated sin θ, cos θ, tan θ) are then defined by the following ratios:



$$\sin\theta \equiv \frac{\textit{opposite side}}{\textit{hypotenuse}} \equiv \frac{OPP}{HYP}$$

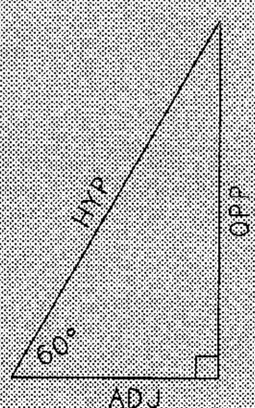
$$\cos\theta \equiv \frac{\textit{adjacent side}}{\textit{hypotenuse}} \equiv \frac{ADJ}{HYP}$$

$$\tan\theta \equiv \frac{\textit{opposite side}}{\textit{adjacent side}} \equiv \frac{OPP}{ADJ}$$

Since the trigonometric functions can be defined as ratios of lengths, they have no units; they are dimensionless. Sine and cosine can never be greater in magnitude than 1, since the hypotenuse must always be longer than the other legs of a right triangle; however tangent can take on any value.

Values of sine, cosine, and tangent. Generally you can find values of trigonometric functions using your electronic calculator.(If you are using degrees set the DEG/RAD switch appropriately.) However, it is a worthwhile exercise to calculate (and ultimately, remember) the sine, cosine, and tangent of some of the more commonly occurring angles, as in this example:

Find the sine of 60°.



DISCUSSION: Recall that in the 30-60-90 triangle the shorter leg is half the hypotenuse. (This can be seen by realizing that this triangle is half an equilateral triangle.) Thus, replacing the labels shown in the drawing by

a = HYP and $\frac{1}{2}a$ = ADJ,

OPP can be obtained using the Pythagorean theorem as follows:

$$(OPP)^2 + (\tfrac{1}{2}a)^2 = a^2$$

$$OPP = (a^2 - a^2/4)^{1/2} = \sqrt{3}a/2$$

Hence, to three significant figures

$$\sin 60^\circ = OPP/HYP = \sqrt{3}/2 = 0.867$$

Values of some other commonly occurring trigonometric functions, which can be calculated in a similar way (see Drill 10), are listed in the following table. You should realize, however, that although right triangles have been used to calculate the values shown in the table, and although the definitions of sine, cosine, and tangent have been stated in terms of right triangles, trigonometric functions can be used even in problems not involving right triangles.

TABLE — Trig functions of commonly used angles.

θ	sin θ	cos θ	tan θ
30°	1/2	√3/2	1/√3
60°	√3/2	1/2	√3
45°	1/√2	1/√2	1

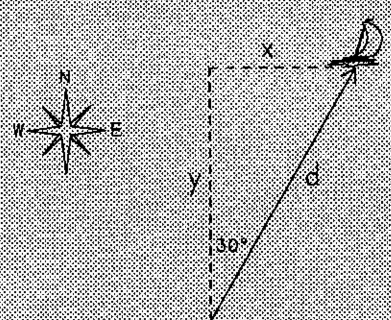
USING TRIGONOMETRIC FUNCTIONS WITH RIGHT TRIANGLES

The most frequent way in which you will use trigonometry in physics problems is to find one side of a right triangular figure, given another side and an angle. For such problems it is useful to remember the meaning of sine and cosine in this form:

$$\text{OPP} = \text{HYP} \sin\theta \qquad \text{ADJ} = \text{HYP} \cos\theta$$

This way of writing the definitions of sine and cosine is used in the first of these examples:

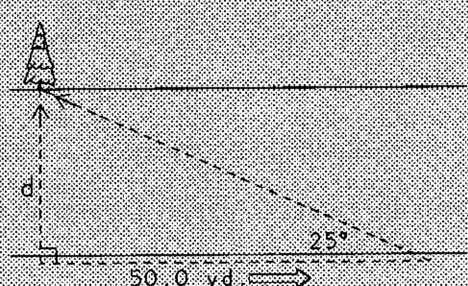
(a) A sailor steers his boat on a direct course 30° east of due North. After sailing 5.0 miles how far to the East and how far to the North has he travelled?



DISCUSSION: Make a diagram. Calling the distance along the direct course d, and the east and north distances x and y respectively, we have

$$x = d \sin 30° = (5.0 \text{ mi})(½)$$
$$= 2.5 \text{ miles.}$$
$$y = d \cos 30° = (5.0 \text{ mi})(\sqrt{3}/2)$$
$$= 4.3 \text{ miles.}$$

(b) On the edge of a river we see a tree directly across from us on the opposite side. Walking 50.0 yards along the edge of the river we again see the tree at an angle of 25° with respect to the river bank. How far away from our starting point is the tree?



DISCUSSION: From this drawing we see that

$$\tan 25° = d/(50.0 \text{ yd}).$$

Solving for d and using a calculator to find tan 25° = 0.468, we have

$$d = (50.0 \text{ yd})(0.468) = 23.3 \text{ yd} .$$

FINDING AN ANGLE GIVEN A TRIG FUNCTION

Inverse trig functions. In the examples given above an angle was the known quantity; the task in both cases was to calculate a side of a right triangle. The reverse problem is also important: given lengths of sides of a right triangle find one of the angles.

Suppose, for example, we know both the hypotenuse (HYP) and the side opposite the required angle θ (OPP), so that we can calculate the ratio $x = OPP/HYP \equiv \sin\theta$. The operation which takes us from x to its corresponding angle θ is called finding the "inverse sine" or the "arcsine" of x. This is written

$$\theta = \sin^{-1}x = \arcsin x.$$

This operation may be carried out using an electronic calculator by entering a value of x and pressing the appropriate function key. Likewise the calculator can be used to find the inverse cosine (arccosine)

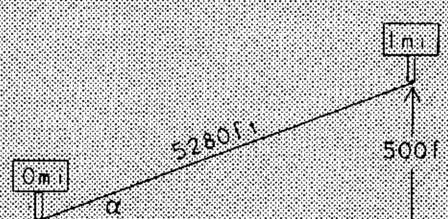
$$\theta = \cos^{-1}y = \arccos y \text{ , where y is the cosine of } \theta.$$

or the inverse tangent (arctangent)

$$\theta = \tan^{-1}z = \arctan z \text{ , where z is the tangent of } \theta.$$

Application using a right triangle. Consider this example:

A car is driving along a straight inclined section of highway. When the mileage markers at the roadside indicate that it has travelled 1.00 mile the gain in elevation of the car is 500 ft. What is the angle of the incline with respect to the horizontal?



DISCUSSION: Referring to the drawing (not to scale), the angle we are asked to find is α. Thus, in terms of the usual notation for labelling a right triangle OPP = 500 ft and HYP = 1.00 mi = 5280 ft . Thus

$$\alpha = \arcsin(500/5280) = \arcsin(0.0947) = 5.43^\circ .$$

Significant figures. In the example above the answer is rounded off to three significant figures. This is done because (as you can verify with a calculator) a change in the value of the sine (0.0947) by one unit in the last place affects the value of the arcsine by approximately 0.01°. In general, there is no simple way to determine the correct number of significant figures in an angle. But in most cases the context of the problem makes it clear how many places are reasonable.