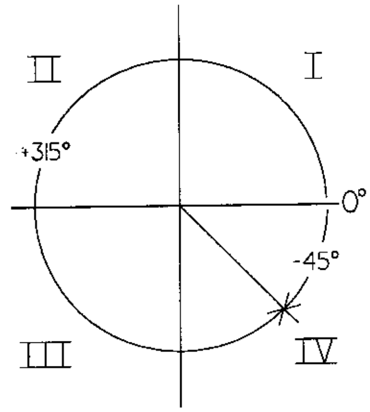
Review 11 — Trigonometry II - Other Triangles and Applications

The examples given in the previous review used right triangles only. In this section, triangles of a general shape are considered, as well as the idea of trigonometric functions of angles which are not positive or acute. Force, as a concept for which trigonometry is useful, is introduced.

TRIG FUNCTIONS OF NON-ACUTE ANGLES

Quadrants and angle convention. It is convenient to picture the entire range of angles as being generated by a straight line sweeping around through the four quadrants of a plane pictured at the right. By convention



- *angles from 0° to 360° are measured* <u>*counterclockwise*</u> *starting from the positive horizontal axis, and*

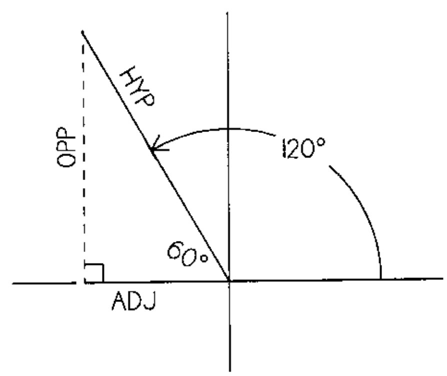
- *angles from 0° to -360° are measured* <u>*clockwise*</u> *starting from the same axis.*

Thus angles between 0° and 90° (acute angles) are said to be in the "first quadrant"or Quadrant I, angles between 90° and 180° are in Quadrant II, etc.

Any given direction can be specified by either a negative or a positive angle. Negative angles may sometimes be a simpler choice, especially in Quadrant IV. For example, the orientation of a line which is 45° *below* the horizontal positive axis may be described either by $\theta = -45°$ or by $\theta = +315°$.

Sine, cosine, tangent. The sine, cosine, and tangent of angles in Quadrants II through IV are defined much like they are defined for angles in the first quadrant. However various trig functions may, in certain quadrants, take on negative values. Here is how the trig functions can be defined in general:

1) Draw a line radially out from the origin pointing in the required direction, then form a right triangle by dropping a perpendicular from the end of the line to the horizontal axis.



As an example, the drawing shows a right triangle formed from a line drawn at an angle of 120°.

2) Except for sign each trig function is defined exactly as before in terms of this triangle, with the vertical side playing the role of OPP, the horizontal side playing the role of ADJ, and the radial line playing the role of HYP.

The labels OPP and ADJ shown in the drawing are appropriate since these sides are opposite and adjacent, respectively, to the vertex at which the 120° angle is located. However, these quantities can take on negative as well as positive values, as follows:

3) The sign of the trig function is determined by assigning a positive value to the vertical side (OPP) if it is in the upper half of the plane, and a negative value if it is in the lower half; likewise ADJ is assigned a positive or negative value depending on whether it is in the right half of the plane (+) or in the left half (-). HYP is always regarded as positive.

Thus, in the example of the trig functions of 120° shown in the drawing, the sine is positive since OPP is positive, whereas cosine is negative since ADJ is negative. The size ("magnitude") of the trig functions are the same as those of the acute angle (60°) at the vertex of the triangle in the center of the drawing. In summary, for this example:

$$\sin 120° = \text{OPP/HYP} = \sin 60° = \sqrt{3}/2$$

$$\cos 120° = \text{ADJ/HYP} = -\cos 60° = -1/2$$

$$\tan 120° = \text{OPP/ADJ} = -\tan 60° = -\sqrt{3}$$

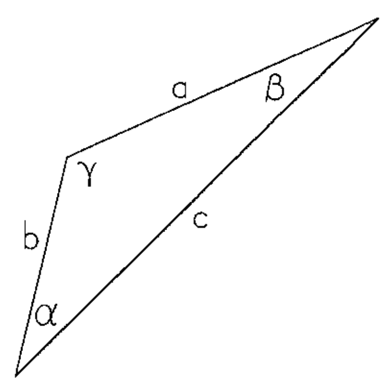
Rather than simply *memorizing* rules about the signs and magnitudes of trig functions in various quadrants it is far more satisfactory to quickly sketch the angle and examine the triangle which is formed by dropping a perpendicular to the axis.

Ambiguity in finding inverse trig functions. The example discussed above illustrates the fact that there is more than one angle corresponding to a given value of a trig function. An electronic calculator returns angle values only between −90° and +90° (Quadrants I and IV) when the $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ keys are used, even though in an occasional problem the desired angle may not lie in the indicated quadrant. Invariably, however, a sketch accompanying the problem will clarify what the correct answer should be—just another reason why a sketch is important in problem solving!

"NON-RIGHT" TRIANGLES

The law of cosines. In an introductory physics course most of the trigonometry which is needed involves right triangles only. Occasionally, however, it may be easier to use one of two rules which relate side lengths and angles in triangles of a general type. The first of these rules, the *law of cosines,* gives a relationship among three sides of a triangle and one of the angles, as follows:

Using the usual notation for sides and corresponding angles, as illustrated here, when the angle involved is α, the law of cosines can be written



$$a^2 = b^2 + c^2 - 2bc \cos \alpha .$$

When either of the other angles is involved the corresponding statements of the law are

$$b^2 = c^2 + a^2 - 2ca \cos \beta$$

or

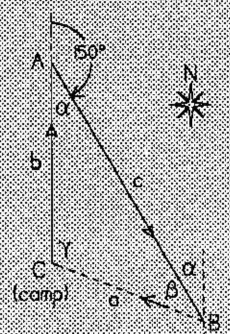
$$c^2 = a^2 + b^2 - 2ab \cos \gamma .$$

The law of cosines is not difficult to remember; it is like the Pythagorean theorem, except for the cross term involving two side lengths and the cosine of the included angle.

This example problem makes use of the law of cosines.

Mr. Smith strolls out into the woods from his fishing camp. He walks a half mile due north, then makes a 150° right turn and walks another 3/4 mile in a straight line. How far is he from his camp?

DISCUSSION: First draw a diagram. The two legs of his route and the line joining his final position and the camp form a general triangle with angle $\alpha = 30°$ at the corner where he changed directions. The distance to the fishing camp is given by



$$a^2 = b^2 + c^2 - 2bc \cos 30°$$

$$= (0.50\text{ mi})^2 + (0.75\text{ mi})^2 - 2(0.50\text{ mi})(0.75\text{ mi})(0.866).$$

Distance to camp, a = 0.40 miles.

In the above example, since a, b, and c are all finally determined, the law of cosines can again be used (in the appropriate form) to solve for each of the unknown angles β and γ. Instead, however, we will find these angles in a follow-up example using another useful rule: the "law of sines."

The law of sines. This rule expresses the remarkable fact that there is a simple direct proportionality between any side of a triangle and the sine of its opposing angle. Using our usual notation for angles and sides of triangles, the law of sines can be stated:

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

The following extension of the previous example uses this rule.

In what direction from his present position (point B in the previous diagram) does Mr. Smith's camp lie?

DISCUSSION: Compass directions are usually stated with respect to the north-south line (dashed line in the drawing). The direction to the camp is thus $\alpha + \beta$. (AC is parallel to the N-S line; hence AB forms an angle α with both these lines.) To find β use the law of sines, as follows:

$$\frac{\sin \beta}{0.50\text{ mi}} = \frac{\sin \alpha}{0.40\text{ mi}}, \quad \text{where } \sin \alpha = \sin 30° = 0.500 .$$

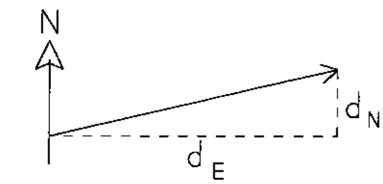
$$\sin \beta = 0.625, \text{ or } \beta = \sin^{-1}(0.625) = 39°.$$

Hence the direction to the camp is $\alpha + \beta = 30° + 39° = 69°$ west of north.

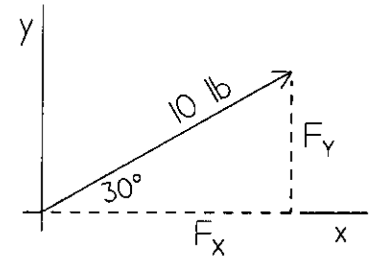
FORCES — ANOTHER APPLICATION FOR TRIGONOMETRY

Forces quantitatively represented by arrows. The direction of a force (a push or a pull) can be depicted in the diagram of a problem by an arrow pointing in the appropriate direction. In addition, in quantitative problems it is useful to draw the length of force arrows in proportion to the strengths of the forces they represent, just as lengths of arrows on a map represent distances travelled. In fact, we can use geometry and trigonometry to analyze the effects of forces in the same way in which certain aspects of motion are analyzed.

Component forces. In some of the examples given in this review and the last one, motion of an object was regarded as a combination of separate motions along different directions. For instance, the track of a ship can be described in terms of the distances the ship moves towards the East and towards the North. In other words, the motion is a combination of an eastward "component" and a northward "component". On a map, these distances are the sides of a right triangle lined up along the east and north directions (d_E and d_N, respectively, in this drawing).



Likewise, it is often fruitful to regard a given force as a combination of component forces—for example, an "x-component" and a "y-component". To find these, the force arrow is drawn as the hypotenuse of a right triangle; the lengths of the sides parallel to the x- and y-directions are proportional to the x- and y-components, respectively. This is illustrated at the right by a force arrow representing a 10 lb force pointing in the designated direction. The components are



$$F_x = (10\ \text{lb})\cos 30^\circ = 8.7\ \text{lb}$$
$$F_y = (10\ \text{lb})\sin 30^\circ = 5.0\ \text{lb}\ .$$