SIMULTANEOUS EQUATIONS

Conditions for a solution. Review 7 considered problems in which more than one unknown quantity could be found using a set of several linear equations. To do this the following requirement must be met:

There must be at least as many independent simultaneous equations as there are unknown variables.

This rule also applies when a set of simultaneous algebraic equations are not linear, although the algebra needed in some cases may be somewhat complicated. Fortunately, almost all instances you are liable to encounter in introductory physics involving squared variables can be handled in a straightforward way using a "substitution of equations" approach.

Simultaneous quadratic and linear equations. When one of the equations is quadratic and others are linear it is relatively easy to eliminate all but one of the variables from the higher power equation. It can then be solved using the usual methods for solving quadratic equations. This is illustrated by the following strictly algebraic example.

Find the values of the two unknowns x and y which simultaneously satisfy the following two equations:

$$y = x^2 + 3x - 6$$
$$y = x + 2$$

DISCUSSION. The value of y from the second equation can be substituted into the first equation to give

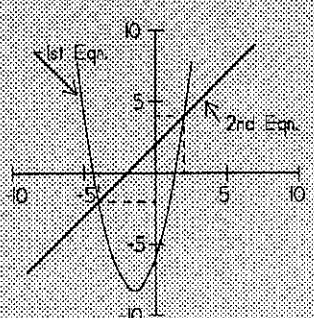
$$x + 2 = x^2 + 3x - 6 ,$$

or

$$0 = x^2 + 2x - 8.$$

The two roots of this equation can be found using the quadratic formula; these are x = 2 and −4. Next, the corresponding values of y are found by substituting each of these values of x into one of the original equations. This gives us two solution pairs, which are

$$(x,y) = (2,4) \text{ and } (-4,-2).$$



(These results may be checked by substituting back into each of the original equations.) Each solution pair corresponds to a point of intersection of the curves which represent the two simultaneous equations. In this example there are two solution pairs, hence two points of intersection, as shown in the graph.

DISCUSSION: Substitute for E to give 16 J = (4 J/cm^2)x^2. Rearranging we have

$$x^2 = 4\ \text{cm}^2 .$$

Taking the square root on both sides of the equation yields the two roots

$$x = +2\ \text{cm (stretch)}$$

and

$$x = -2\ \text{cm (squeeze)}.$$

The quadratic formula. In the last example, the constant B was zero; consequently finding the solutions required no special technique except finding an ordinary square root. However, in general, roots of the equation $0 = Ax^2+Bx+C$ can always be obtained by applying the following "quadratic formula":

$$x = \frac{-B \pm\sqrt{B^2 - 4AC}}{2A}$$

This formula is so general and useful as to be worth memorizing. In the following example illustrating the use of the quadratic formula, units have been left out in order to clarify the mathematical manipulation. Problems which involve units are included in the accompanying drill and are especially emphasized in the next review.

In a certain problem y and x are related by the expression $y-1 = x(4x+2)$. What are the values of x corresponding to $y = 3$?

DISCUSSION: The expression can be rewritten as

$$y = 4x^2 + 2x + 1.$$

Substituting $y=3$, the equation can be rearranged into the standard form

$$0 = 4x^2 + 2x - 2 .$$

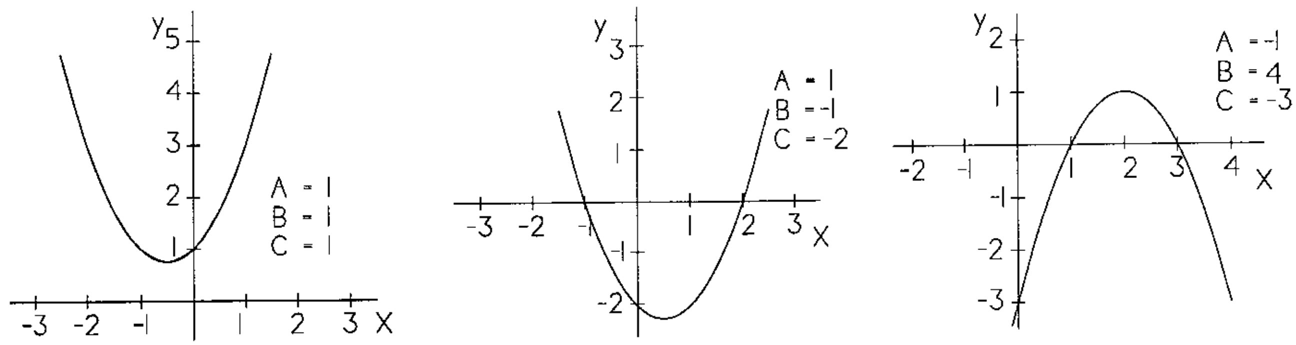
Identifying $A=4$, $B=2$, and $C=-2$, the quadratic formula gives for the two roots

$$x = \frac{-2 \pm\sqrt{4-(4)(4)(-2)}}{(2)(4)}$$

$$= ½ \text{ and } -1 .$$

Two roots are always to be expected, although one of the solutions sometimes is rejected because it is not meaningful physically. Occasionally both roots are identical; graphically such a solution corresponds to a point located at the turning point (bottom of the "bowl") of the parabola.

Other parabolas. When B and C are not both zero, the graphs of the quadratic equation are also parabolic but may be shifted up, down, or sideways, or flipped upside down (but never rotated), depending on the values of A, B, and C. Three cases are illustrated here.



The approximate width of the parabola and its location with respect to the axes may not be readily apparent, given values of A, B, and C, but a couple of facts may occasionally be useful for a quick graphical interpretation of a result: (a) negative values of A give rise to "upside-down" parabolas, and (b) C is the y-intercept (the value of y where the curve crosses the y-axis).

SOLVING QUADRATIC EQUATIONS

Standard form of the equation and roots. Except for the point at the very bottom (or top) of the curves where each of the parabolas shown above turn around, there are *two values* of x corresponding to every possible value of y. Of particular interest are the two values of x where the curve crosses the x-axis. These are the two solutions or "roots" of an expression which has the following form:

$$0 = Ax^2 + Bx + C\ .$$

Sometimes the algebraic statement of a physics problem gives us this special case of a quadratic equation directly. However, in general, a non-zero value of y is one of the given quantities and we must rearrange the equation into the above "standard form" in order to find the values of x which satisfy the problem. Stated briefly:

> *Substitute the known value of y into the quadratic equation which represents the physics. To find the two roots x, it is generally useful to algebraically rearrange the equation into "standard form"* $0 = Ax^2+Bx+C$.

Finding a pair of roots as the solution of a physical problem is illustrated by the following example. (This particular case is simple enough mathematically that it isn't necessary to write the quadratic equation in standard form.)

> The energy of a spring which is stretched or squeezed an amount x (in cm) from its normal length is given by
>
> $$E = (4\ \text{Joules/cm}^2)x^2\ .$$
>
> At what values of x is the energy 16 Joules?
>
> DISCUSSION: (Next page.)

Review 12 — Quadratic Expressions

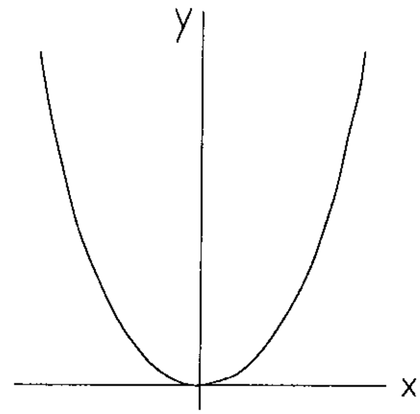
A step up in complexity from linear equations are quadratic equations, "power functions" in which terms occur containing the independent variable raised to the second power. Expressions of this type are of especial importance in mechanics, which is the core subject of physics.

THE EQUATION AND ITS GRAPH

General expression. The most general form of a quadratic equation can be expressed as follows:

$$y = Ax^2 + Bx + C\,.$$

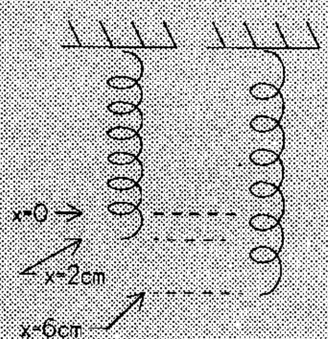
A, B, and C are constants, not depending on x, which determine in detail just how y will change as x takes on different values.



Squared proportion. In the simplest cases, both B and C equal zero. When the resulting equation

$$y = Ax^2$$

is plotted on a graph a symmetrical bowl-shaped curve called a *parabola* results. (For a negative value of A, the bowl is upside-down.) This equation represents a type of proportion; not a linear proportion, but a squared (or quadratic) proportion. But as with linear proportions we can often use ratios to find answers to problems, as in this example:



The energy stored in a simple helical spring depends on the amount x which the spring is stretched or compressed from its normal length. The relationship is

$$\text{Energy stored} = \tfrac{1}{2}kx^2$$

where the value of k, called the spring constant, depends on how the spring is constructed. If 10 Joules (J) of energy is stored when a certain spring is extended 2.0 cm, how much energy is stored when the stretch is increased to 6.0 cm?

DISCUSSION: The stored energy is proportional to the square of the stretch Therefore the ratio of the energies is equal to the ratio of squared stretches, as follows:

$$\frac{\text{Energy}(x=6\text{ cm})}{\text{Energy}(x=2\text{ cm})} = \frac{(6.0\text{ cm})^2}{(2.0\text{ cm})^2}\,.$$

Thus setting Energy(x=2 cm) = 10 J and rearranging, we have

$$\text{Energy}(x=6\text{ cm}) = (10\text{ J})(36/4) = 90\text{ Joules.}$$