Review 14 — Other Power Laws

Some important physics is described by power functions other than linear and quadratic. Especially important are expressions involving negative powers—inverse functions—since such relationships often apply to phenomena in which the increase in one quantity is accompanied by a decrease in another quantity. This review deals with these and other power functions which come up in a course in introductory physics.

CUBICS AND QUARTICS

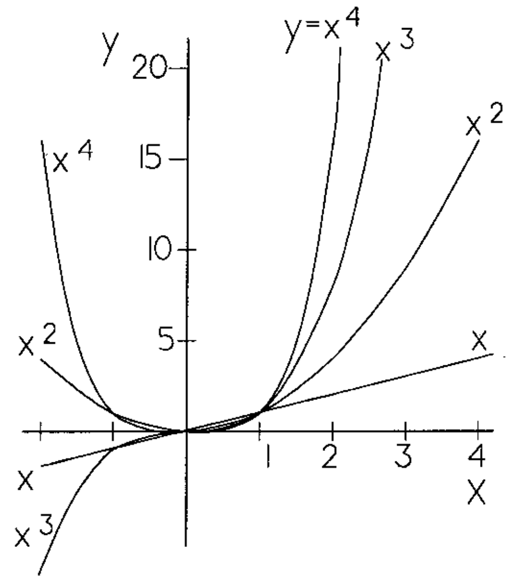
Power laws and ratios. The graphs of the power functions

$$y = Ax^3 \text{ and } y = Ax^4$$

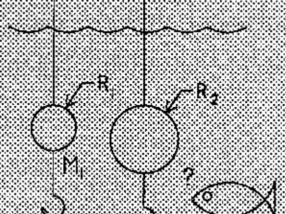
are similar in general appearance (at least for positive x) to the quadratic proportion

$$y = Ax^2.$$

Each of these describes an increasingly steep monotonic rise in the value of y as x increases. Cubic (3rd power) and quartic (4th power) terms arise occasionally in introductory physics equations, but almost always as the only power term in the equation. In such cases the use of ratios in solving problems is a possibility, as in this example:



A spherical lead fishing weight of radius 0.50 cm has a mass of 5.9 gm. How much lead is in a similar weight of radius 0.75 cm?



DISCUSSION. The amount of material in each weight is proportional to the volume which, for a sphere, goes as the cube of the radius R. (Specifically the formula is $V=(4/3)\pi R^3$.) Hence the ratio of the masses is the same as the ratio of the volumes which, in turn, is the same as the ratio of the radii cubed.

Using subscript 1 and 2 for the smaller and larger spheres respectively, we have

$$Mass_2 / Mass_1 = V_2 / V_1 = R_2^3/R_1^3.$$

Thus

$$Mass_2 = Mass_1(R_2/R_1)^3 = (5.9 \text{ gm})(1.5)^3 = 20 \text{ gm}.$$

Solving problems using ratios should be considered whenever a physical quantity is known to be proportional to a power of some other quantity.

Thus for problems involving a simple "power law" $y = kx^n$, we can often find an answer even if the value of the coefficient k is not known.

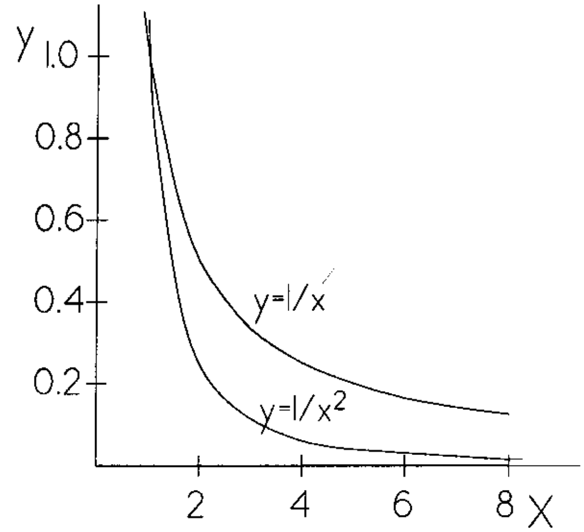
INVERSE POWERS

In several places in physics you will come across functions whose form resembles these equations:

$$y = kx^{-1} = k/x$$

or

$$y = kx^{-2} = k/x^2 .$$



A major feature of such "inverse proportions" is that quantity y approaches zero as the quantity x gets extremely large. Likewise for smaller and smaller values of x the size of y increases without limit. This type of behavior is illustrated by the graphs on the right.

Simple inverse proportion. The example below, which is familiar to most students from their study of chemistry, has to do with the behavior of gases under ordinary conditions.

Boyle's law tells us that under most conditions if we change the volume V of a sample of gas while keeping the temperature constant, the pressure p of the gas will vary inversely as V. This can be expressed

$$p = k/V .$$

Suppose some ordinary room air (at atmospheric pressure p_a) is forced into a balloon where it occupies one third the volume it had outside the balloon. What is the pressure p_b inside the balloon compared to atmospheric pressure?

DISCUSSION. Set up the following ratio relationship:

$$\frac{p_b}{p_a} = \frac{k/v_b}{k/V_a} = \frac{V_a}{V_b} = \frac{V_a}{V_a/3} = 3$$

or

$$P_b = 3\,P_a.$$

The pressure in the balloon is 3 times atmospheric pressure.

Inverse square laws. Phenomena which can be described by an equation of the type $y=kx^{-2}$ occur in several places in physics, almost always connected with some effect which decreases uniformly in all directions as the distance from a central point is increased. For example, the light intensity (a measure of the energy of the light illuminating an object) decreases as the inverse square of the distance from a light bulb. This phenomenon is used in the next example problem.

A lamp emits light equally in all directions. The intensity of the light illuminating an object at a distance r is given by $I = F/r^2$, where F is a constant related to the total energy of the light given off by the bulb. If the intensity reaching an object placed at 0.30 meters is 100 lux, what is the intensity at 1.0 meters?

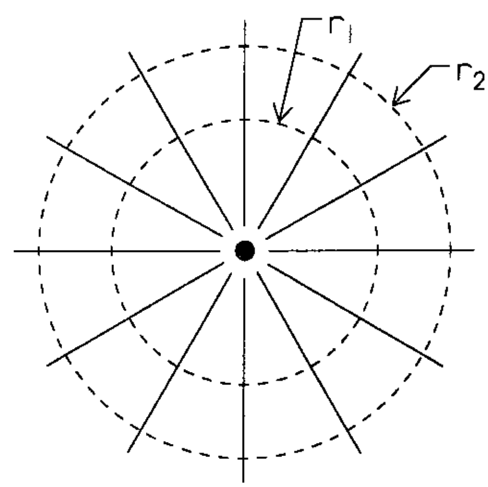
DISCUSSION: To solve the problem it is not necessary to know the value of F, or even how the units of I are defined. The ratio expression is

$$\frac{\text{I at 1.0 meter}}{\text{I at 0.30 meter}} = \frac{(0.30\ \text{m})^2}{(1.0\ \text{m})^2}$$

which can be rearranged to give the answer:

$$\begin{aligned}\text{Intensity at 1.0 meter} &= (100\ \text{lux})(0.30/1.0)^2 \\ &= 9.0\ \text{lux.}\end{aligned}$$

A "geometrical" interpretation of the inverse square. Electrical forces due to bits of electrical charge or gravitational forces due to concentrations of mass are two other situations in which some quantity varies as the inverse square of the distance from a central point. Such phenomena lend themselves to a common interpretation which is useful for visualizing the physics. In this "geometrical" interpretation we imagine that the central point "radiates" its influence in the form of a equally spaced straight lines emerging radially. Consider several concentric spheres located at distances r_1, r_2, etc. from the central point, as shown in this drawing. Since the surface areas of the spheres are larger the greater their radii, the number of lines that cross a given area on each sphere decreases with radius, i.e., the lines become less densely packed with distance. In fact since the surface areas increase as r^2, the "density" of the lines crossing each sphere shown in the drawing is proportional to $1/r_1^2$, $1/r_2^2$, etc.Thus we can picture "inverse-square phenomena" this way:

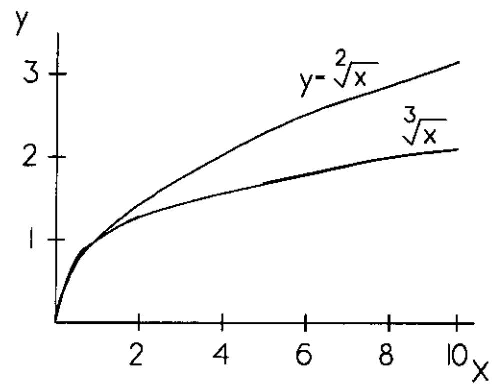


• *a quantity which varies as the inverse square of the distance from a point is often represented by imaginary lines directed uniformly outward from the point; the density of these lines is proportional to the quantity at that distance.*

The lines are given different names when this idea is used in different parts of physics. For instance, in the example of light intensity given in the last example problem, the lines are often called "rays"; in electrical problems they are called "field lines".

FRACTIONAL POWERS

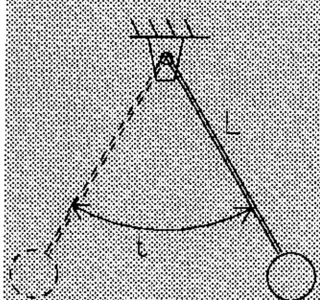
Powers and roots. Raising a number to a rational fraction (½, ⅓, etc.) is the reverse operation to taking an integer power. For example, squaring a square root gives back the original number, i.e., $(\sqrt{x})^2 = (x^{1/2})^2 = x$. (Note that the word "root" is used in a somewhat different sense than when it is used to denote the solutions of an equation.) Thus it is not surprising that the graphs of the square root and cube root shown on the right contrast strongly with graphs of integer power functions. Instead of more and more rapid increases of y as x gets larger, the increases in these functions "slow down" at large x.



An example using ratios. The following is an example taken from physics which involves a fractional power function.

The time τ which a pendulum takes to make a complete swing back and forth is given by the following expression:

$$\tau = 2\pi\sqrt{\frac{L}{g}}$$



In this equation L is the length of the thin rod supporting the pendulum bob and g is the acceleration due to gravity. (Acceleration of gravity is discussed in Review 13.)

If a certain pendulum makes one swing in 1.0 s, how long will it take to make a swing if the supporting rod is shortened to half its original length?

DISCUSSION: The ratio approach eliminates the need to know most of the quantities in the equation. Thus we can write

$$\frac{\tau(\text{final})}{\tau(\text{original})} = \frac{\sqrt{L/2}}{\sqrt{L}}$$

$$\tau(\text{final}) = (1.0\ \text{s})\sqrt{\frac{1}{2}} = 0.71\ \text{seconds.}$$