# Review 15 — Sinusoidal Functions - Oscillating Phenomena

Much of what we casually observe around us changes in a cyclic or repeating way: the coming and going of seasons, the rise and fall of water along the edge of a beach, the beating of a butterfly's wings. Periodic oscillations are of particular importance in the realm of very rapid and microscopic changes and are intimately connected with moving waves: sound consists of vibrations in the air and light depends on the transmission of periodically varying electric and magnetic forces. Fundamental to the quantitative treatment of such phenomena is the use of sinusoidal functions, expressions in which the independent variable is contained in the argument of a sine or cosine. This Review goes over basic ideas needed to use these functions in connection with oscillations and waves, and also discusses radian angle measure and its applications.

## SINUSOIDAL FUNCTIONS

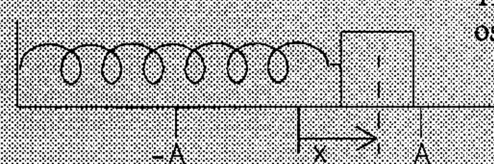
**Describing oscillations mathematically.** Many important physical systems –such as bouncing springs or swinging pendula – are observed to oscillate in a way which can be simply described using the sine and cosine functions first encountered in your study of trigonometry. Even systems which exhibit more complicated repetitive motion can be analyzed using expressions which are combinations of simple sine and cosine functions.

In trigonometry the quantity $\theta$ (the "argument") in $\sin \theta$ and $\cos \theta$ is usually regarded as a dimensionless *constant*, representing some fixed direction in terms of degrees. When used for describing oscillating behavior the argument instead *varies* uniformly with time t: $\theta = kt$. As t increases the argument goes progressively from 0° through all intermediate values to 360° ( which is equivalent to 0°) and continues to repeat this sequence; the corresponding values of sine and cosine oscillate continuously between +1 and -1. Thus the expressions

$$x = A \cos kt \quad \textit{and} \quad y = A \sin kt$$

both represent quantities which vary continuously and periodically between A and -A. The units of x and y are the same as that of A; if t is measured in seconds, k is in deg/sec. It is also common practice to express the argument in radians, as explained later in this Review, but for the time being we will use degrees.

The following problem uses the "spring-mass oscillator" discussed in the previous Essay, as an example of an oscillating system whose motion can be described using a sinusoidal function.




The position x of a mass attached to a spring ("spring-mass oscillator") is found to vary according to

$$x = A \cos kt$$

where A = 1.5 cm and k = 900 deg/sec.

How long does it take for the mass to make one complete oscillation (back and forth to the starting position)? What is the position x when 1/8 of this time has elapsed, starting at t = 0?

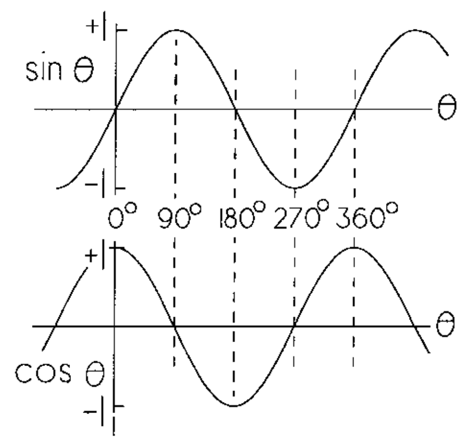
DISCUSSION: A complete oscillation requires the argument of the cosine to change by 360°, thus kt = 360°, or

$$t = 360°/k = 0.40 \text{ seconds.}$$

One eighth of a full oscillation takes (0.40 sec)/8 = 0.050 sec. The argument of the cosine is kt = (900 deg/sec)(0.050 sec) = 45°. Thus

$$x = (1.5 \text{ cm}) \cos 45° = 1.1 \text{ cm}.$$

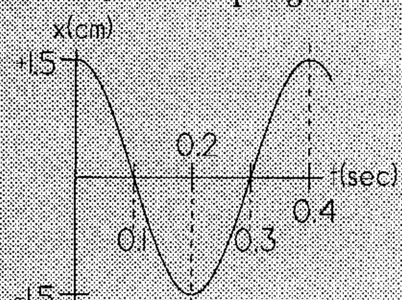
**Curves of sine and cosine.** Sin θ and cos θ versus θ are plotted one under the other at the right. Both have the same "sinuous" shape, but are shifted relative to one another along the θ-axis by 90°. You should be able readily to sketch these curves.




*Sin θ and cos θ vary continuously between the extreme values ±1 at intervals of θ = 360°, crossing through the axis with a finite slope; sin θ rises at θ = 0 from zero towards a maximum, whereas cos θ has a maximum at θ = 0.*

These guidelines are applied in the following example exercise:

On a graph of position x versus time t sketch a curve showing the motion (for one complete oscillation) of the "spring-mass oscillator" described in the last example.



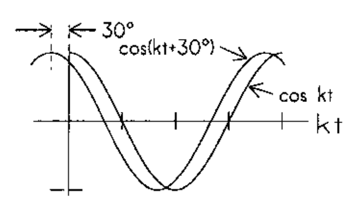

DISCUSSION: The motion is described by a cosine function:

$$x = (1.5 \text{ cm}) \cos (900°/\text{sec})t .$$

At t=0 the curve has a maximum x = 1.5 cm. It returns to that value when the argument (900 °/sec)t = 360°, i.e., when t = 0.40 sec, as shown.

**Phase angle.** Whether a sinusoidal oscillation is to be described by cos kt or sin kt clearly depends on whether t = 0 is chosen to be at the maximum or at the zero-point of the oscillation. A more general choice for t = 0 can be expressed by including a constant $\theta_o$, called a *phase angle,* in the argument of the function. Thus, the curves of the functions

$$x = A \cos(kt+\theta_o) \quad and \quad y = A \sin(kt+\theta_o)$$




resemble the cos kt and sin kt curves, but are shifted along the kt-axis by an amount $\theta_o$. A cosine curve with $\theta_o$ = +30° is pictured here.

*A positive phase angle $\theta_o$ shifts the sinusoidal curve by an amount $\theta_o$ towards lower values of kt; a negative phase angle shifts the curve towards higher values.*

**Trigonometric identities.** Not all repeating phenomena can be described by simple sine or cosine functions. However, at least in principle, all periodic functions can be formed from combinations of simple sines or cosines. Some especially important cases require the combining of just two sinusoidal functions into a more valuable expression, often in a form in which the several arguments are joined together.

A number of formulas (trig identities) for combining two or more sinusoidal functions have been worked out and are listed in trigonometry text books. If such mathematics is needed in a physics problem, the identity will usually be given or you will be able to look it up. It is mostly important that you be aware that the identities exist. Here are a couple of examples:

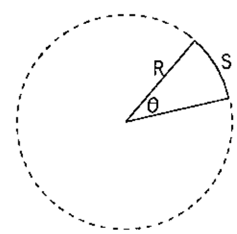
$$\sin\theta + \sin\phi = 2\sin \tfrac{1}{2}(\theta + \phi)\ \ \cos\tfrac{1}{2}(\theta - \phi)$$

$$\sin\theta \cdot \cos\phi = \tfrac{1}{2}\sin(\theta + \phi) + \tfrac{1}{2}\sin(\theta - \phi)$$

## RADIAN MEASURE

**Arcs and angles.** In the analysis of periodic phenomena using radians as the units of angle measure, rather than degrees, generally leads to simpler formulas. This is because the size of a degree is based on an arbitrary choice, viz., the choice of 360° to represent one complete rotation. Radian measure is a more natural system, based on the geometry of a circle, as follows:

*The size of an angle, measured in radians, is the ratio of the length of the circular arc subtended by that angle to the distance to the arc.*




In terms of the diagram at the right

$$\theta = S/R\ ,\ \theta \textit{ in radians.}$$

Whereas radian measure is often preferred in the treatment of oscillating systems, degrees are more often used to describe direction (especially in experimental situations). However, there is some value in being able to use either system of measurement in both types of work. For instance, in a situation in which an arc length is of importance, radians may be favored since the simple formula $S = R\theta$ can be used.

**Interchanging radians and degrees.** Many scientific calculators have DEG/RAD conversion functions. But apart from that, there is little need to memorize a conversion factor; one need only remember that a full circle is an arc of length $2\pi R$, and corresponds to $(2\pi R)/R = 2\pi$ radians. Thus the direct proportion relating the units can be stated

- *angle in radians is to angle in degrees as 2π is to 360.*

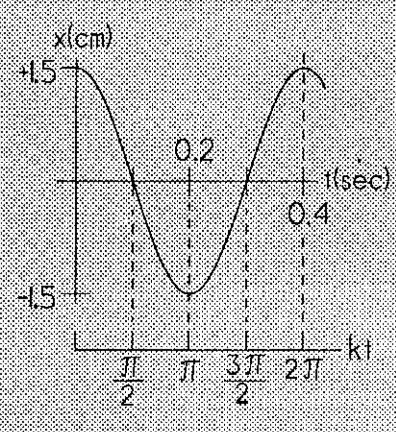
Certain cardinal point conversions are important enough to remember outright. As illustrated in the following problem, these include

$$45° = \pi/4 \text{ radians} \qquad 90° = \pi/2 \text{ radians}$$
$$180° = \pi \text{ radians} \qquad 270° = 3\pi/2 \text{ radians}\ .$$

These angles are used in labeling the drawing in the following problem.

Write an equation for the motion of the spring-mass oscillator described in the previous examples using radians. Sketch a graph of the motion and label several significant points on the time axis with the corresponding values of radians.

DISCUSSION: The coefficient k in the argument is given in terms of radian measure as



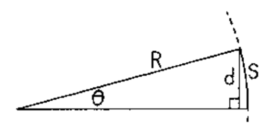

$$k = \left(900\ \frac{\text{deg}}{\text{rad}}\right)\left(\frac{2\pi\ \text{rad}}{360\ \text{deg}}\right) = 5\pi\ \frac{\text{rad}}{\text{sec}}$$

The motion is thus given by

$$x = (1.5\ \text{cm})\cos 5\pi t$$

where, according to common practice, units are left out of the argument. (Usually, the appearance of $\pi$ explicitly in the argument signals that radians is the unit.) In the drawing, the points chosen for labelling are maxima, minima, and axis crossing points (corresponding to the spring at its unextended length.)

**Small angle approximations.** Angle $\theta$ in the right triangle pictured here is small compared with a right angle. The arc S subtended by $\theta$ at distance R is drawn in for comparison with the side of the triangle d opposite to $\theta$. The two lines are nearly indistinguishable: $S \simeq d$. As an approximation, we can use S instead of d in the definitions of the trig functions of $\theta$. For instance, $\sin\theta = d/R \simeq S/R = \theta R/R = \theta$, which is the angle itself (providing $\theta$ is expressed in radians.) Expressions for other trig functions can likewise be simplified.

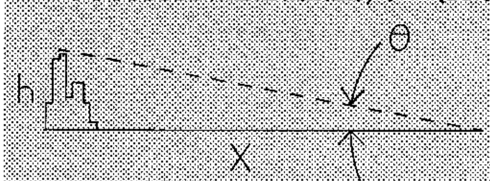



*For small angles $\theta$, when $\theta$ is expressed in radians: $\sin\theta \simeq \theta$, $\tan\theta \simeq \theta$, and $\cos\theta \simeq 1$.*

These approximations are extremely good for angles of the order of degrees. For example, for $\theta = 5° = 0.09$ rad, $\sin\theta$ differs from the radian value by only about 0.1% .

A building viewed at a distance of 0.50 miles, subtends an angle of 11.0°. Use a small angle approximation to estimate the building's height. Compare this result with the more exact value obtained using a trig function.

DISCUSSION: In radians, $\theta = (11.0°)(2\pi\ \text{rad}/360°) = 0.192$ rad.




Using the small angle approximation

$$h/x = \tan\theta \simeq \theta = 0.192$$

or

$$h \simeq (0.192)(0.50\ \text{mi})(5280\ \text{ft/mi}) = 507\ \text{ft}.$$

More exactly, $h = x\tan 11° = 513$ ft, about 1.2% greater than the approximate answer.