Review 16 — Exponential Functions - Growth and Decay

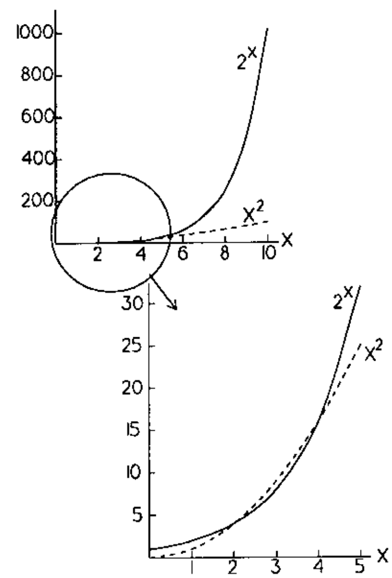
The world abounds in situations in which some quantity rapidly grows in size– not only without apparent limit– but with ever increasing rapidity. We hear of runaway nuclear reactions or explosive population growth– phenomena which almost imply a lack of control. These are examples in which growth, in fact, feeds upon itself– the larger the quantity, the larger the rate at which it increases. Most of these situations, especially in physics, can be described using an exponential function like a^{bt} (t being time). Another group of important occurrences are described by the related exponential decay function a^{-bt}; in such situations quantities fade away towards zero at an ever declining rate. This Review deals with the essential mathematics needed to apply these functions to phenomena discussed in physics courses.

EXPONENTIAL GROWTH

The exponential function. The function of x

$$y = a^{bx}$$

is an exponential function with "base a". The exponent bx, must be dimensionless.



It is instructive to compare a function of this type, for instance 2^x (base 2), with a corresponding power function (x^2) for positive values of x. The two functions are plotted at the right; an expanded portion of the curves (at small x) is also shown. Despite a rapid rise of x^2 at smaller values of x, the function 2^x ultimately catches up and surpasses it. This characteristic behavior holds no matter what the base or power; the exponential function ultimately will overwhelm any power function. This is important because many physical phenomena meet the criterion for exponential growth, as follows:

> *A quantity for which the rate of growth is proportional to the quantity itself follows an exponential growth curve.*

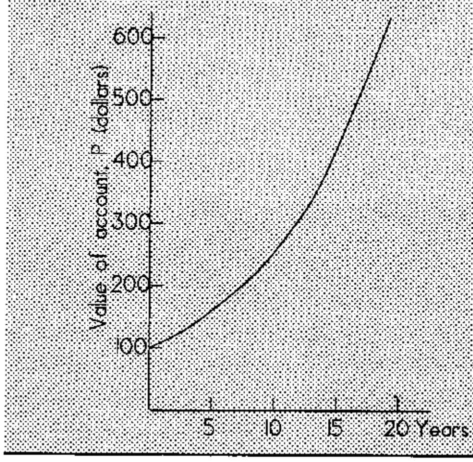
A non-physical example, compound interest, illustrates this idea:

A certain money account is credited with 10% interest every January 1. If this account starts out with a principal P_o, develop an expression for the principal P_t after t years. If P_o = \$100, what is the account worth, assuming no withdrawals, after 10 years? –after 100 years?

DISCUSSION: After 1 year the account is worth $P_1 = P_o + 0.1P_o = P_o(1.1)$. After two years $P_2 = P_o(1.1) + (0.1)P_o(1.1) = P_o(1.1)^2$. Extending the reasoning to t years we get

$$P_t = P_o(1.1)^t\,.$$

This is an exponential growth equation with base (1.1). (The exponent, which is the number of times the base is to be multiplied by itself, must be dimensionless. Thus t should be interpreted as "number of interest periods" = time in years ÷ time in an interest period.)



After 10 years

$$P_{10} = (\$100)(1.1)^{10} = \$259 .$$

After 100 years

$$P_{100} = (\$100)(1.1)^{100} = \$1{,}378{,}061.$$

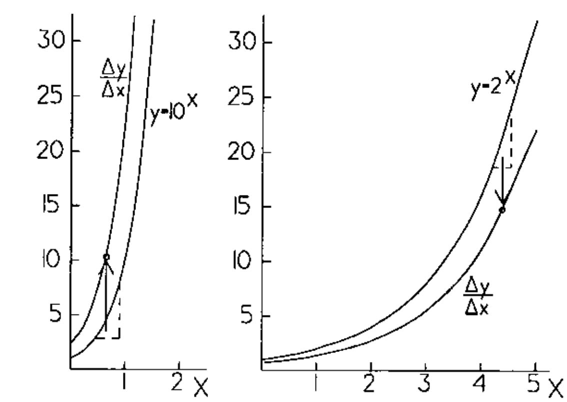
Thus, even for a relatively weak exponential function (base close to 1), the potential for growth is inexorable, given sufficient time. The account values for the first 20 interest periods, joined by a smooth curve, are plotted at the left.

In this example the exponential is interpreted in its most basic way, as an integer number of multiplications of the base by itself. It requires more advanced mathematics to understand the meaning of the base raised to a general non-integer power. For practical purposes, however, it is sufficient to think of the general exponential function as the numbers along the smooth, continuous curve joining integer powers of base a. These values can be found using electronic calculators or computers.

Choice of a base. Certain bases lend themselves well to certain applications, even though an exponential expressed using one base can be rewritten in terms of any other base (see below).

For example, 2^x is handy for describing quantities which double in given intervals of time and for the discussion of probabilities. Moreover, integer values of x generate the numbers 2, 4, 8.... of the binary counting system used in computers. Base 10 is also convenient; for integer values of x, 10^x yields the place values 10, 100, 1000... of our usual decimal counting system. There is, however, still another base lying between 2 and 10, which is often favored because it generally gives rise to simpler expressions in mathematical derivations involving exponential functions. This is the "natural base" e = 2.7182..., discussed below.

Natural base e. For any exponentially varying quantity $y = a^{bx}$, the rate at which y increases is proportional to y itself; the proportionality constant depends on the choice of base. The rate of increase can be estimated from a curve of y versus x; at any point on the curve the rate is approximately rise ÷ run = $\Delta y/\Delta x$. For example, the curve of 10^x and the corresponding curve of $\Delta y/\Delta x$ versus x are plotted at the right; a similar pair of curves for 2^x is also shown. In one case the rate exceeds the function itself; in the other case the rate is smaller than the function. On the other hand, for the natural base e, the curves of $y = e^x$ and $\Delta y/\Delta x$ versus x are identical; the proportionality constant linking the two quantities is 1. This fact accounts for the mathematical simplicity inherent in using e as the base of the exponential function.



For the natural base e = 2.7182..., the rate of increase of e^x equals e^x itself.

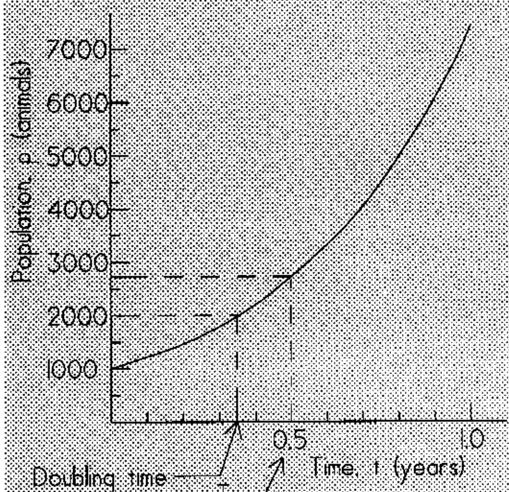
The following example explores some of the salient features of the mathematical description of exponential growth which uses the natural base.

Suppose a population of animals p increases exponentially according to

$$p = p_o\, e^{t/T},$$

where t is in years, p_o = 1000 animals, and T = 0.50 years. Plot p as a function of t out to 1 year. What is the significance of the constant p_o? –of the constant T? From the curve estimate the time it takes for the population to double and compare this with T.

DISCUSSION: The e^x key on your calculator can be used to plot the curve shown here. An example calculation is:



At t = 0.30 years

$$p = (1000 \text{ animals})e^{0.60}$$
$$= 1822 \text{ animals.}$$

The curve intersects the vertical axis at p_o = 1000 animals, which is the initial population (at time t = 0).

T is the length of time for the population to increase by a factor e, i.e., to 2718 animals. This *characteristic time* is somewhat greater than the *doubling time*, which, from the graph, is estimated to be about 0.35 years.

Transformation of bases. The example problem given above asks about "doubling time", which is sometimes a desirable way to think about certain exponential growth phenomena; during any period equal to a doubling time the quantity increases by a factor of 2. In such cases it may be preferable to describe growth using base 2, this way:

$$y = y_o\, 2^{t/T_2}$$

where T_2 is the doubling time. T_2 is proportional to T, the "characteristic time" used in the exponential function with base e. (See the example above.) In fact

- *any base can be used to describe exponential growth; the exponents used with the various bases are proportional to one another.*

The following formulas relating growth functions using different bases can be derived using logarithms. (See Review 17.)

$$2^x = e^{0.69\,x} \qquad 2^x = 10^{0.30\,x}$$

$$10^x = e^{2.3\,x} \qquad e^x = 10^{0.43\,x}$$

EXPONENTIAL DECAY

Just as there are cases of a quantity whose rate of growth is proportional to the quantity itself, a large number of phenomena exhibit *rates of decrease* of a quantity which are proportional to the quantity. These cases of *"exponential decay"* are most often written in terms of the natural base e; the exponent usually contains an explicit minus sign.

> *A quantity for which the rate of decrease is equal to the quantity itself is described by an exponential decay function* $y = y_0\, e^{-x}$.

As in the case of exponential growth, the exponent is dimensionless, and is often written as a ratio t/T, where T is often called the " time constant".

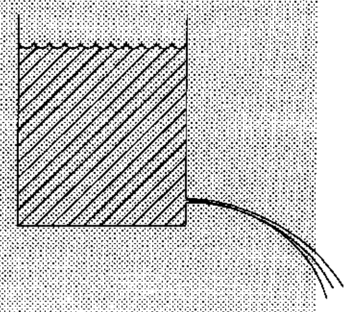
> *During one time constant an exponentially decaying quantity is reduced to 1/e (37%) of its former value.*

A graph typical of exponential decay is examined in the Discussion part of the following example problem.

Water is flowing from a small hole in the bottom of a cylindrical tank. The rate of flow at any time depends on the volume V remaining in the tank, such that

$$V = V_0\, e^{-t/T},$$

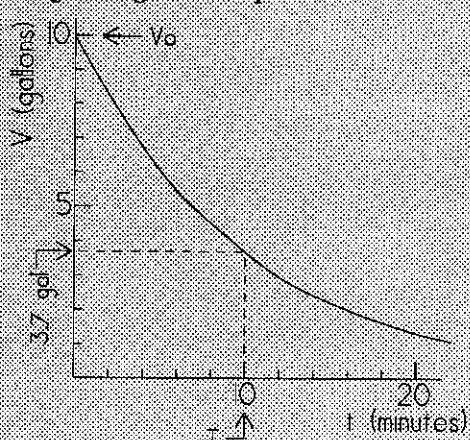
where V_0 = 10 gallons and T = 10 minutes.



What is the significance of V_0? –of T? Plot V versus t out to 20 minutes and mark the positions of $V = V_0$ and t = T.

Assuming the exponential decrease in volume holds true at all times, how much water remains in the tank after 20 minutes? – after 30 minutes?

DISCUSSION: V_0 is the initial volume of water in the tank (at t = 0). T, the time constant of the decay, is the period of time it takes for the volume to decline to 0.37 of its value at the beginning of that period.



Thus, at t = T = 10 minutes

$$V = (0.37)(10\text{ gal}) = 3.7\text{ gal}.$$

Likewise, after another 10 minutes, i.e., at t = 2T

$$V = (0.37)(3.7\text{ gal}) = 1.4\text{ gal}.$$

Finally, after another time interval T (at t = 30 minutes)

$$V = (0.37)(1.4\text{ gal}) = 0.5\text{ gal}.$$
