Review 17 — Logarithmic Functions - Big Things, Little Things

Logarithmic functions are the inverse of exponential functions. For example, $\log(10^x) = x$; likewise, $10^{\log x} = x$. In effect, the logarithm of a number is simply the *exponent* to which a base can be raised to give that number. Just as scientific notation makes use of exponents to write very large or very small numbers, so some scientific quantities which vary over a large range are conveniently described in terms of the exponents themselves, i.e., in terms of logarithms. This section reviews how logarithms are manipulated mathematically, and how they are used to scale and graph physical information.

LOGARITHMS TO THE BASE 10

Logs and powers of 10. A good sense of the meaning of logarithms comes from considering how numbers are written in scientific notation. For example the numbers

$$0.01 \qquad 1 \qquad 100$$

can be expressed as follows:

$$10^{-2} \qquad 10^{0} \qquad 10^{2}\,.$$

The *logarithms to the base 10* (denoted by log_{10}, or simply *log*) of these numbers are the exponents themselves. Thus

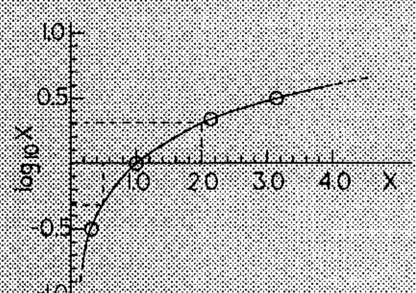
$$\log_{10} 0.01 = -2 \qquad \log_{10} 1 = 0 \qquad \log_{10} 100 = 2\,.$$

This concept of a logarithm as an exponent can be extended to include non-integer powers as well. In general

- *the logarithm to the base 10 of a number is the exponent in the power of ten which yields that number, i.e.,* $\log_{10} 10^x \equiv \log 10^x = x$.

As suggested in the previous review, the use of exponents which are integers or rational fractions are readily understood, whereas general non-integer exponents require more sophisticated explanation. For simplicity, however, non-integer exponents (logs) can be thought of as values along a continuous curve connecting integer (or fractional) powers. This approach is applied in the following example, in which a typical curve of the logarithmic function, including non-integer values, is plotted.

Plot a curve of log x ($\log_{10} x$) versus x by connecting points corresponding to $x = 10^{1/2}, 10^{1/3}$, 1, and $10^{-1/2}$, with a smooth curve. From the curve estimate log 2 and log ½, and compare with values obtained from an electronic calculator.



DISCUSSION: Using the basic property $\log 10^x = x$, we obtain:

$$\log 10^{1/2} = \log 3.16 = 1/2$$
$$\log 10^{1/3} = \log 2.15 = 1/3$$
$$\log 10^{0} = \log 1 = 0$$
$$\log 10^{-1/2} = \log 0.316 = -1/2\,.$$

The corresponding points are plotted above. From the smooth curve connecting the points we estimate log 2 = 0.30 and log ½ = -0.30 . A calculator (which uses a sophisticated computational algorithm) yields ±0.30103 for these logarithms.

Some important particulars. The example just discussed highlights these facts:

> *Logarithms of numbers in the range 0 to 1 are negative, and log 1 = 0. Also, logarithms are not defined for numbers less than or equal to zero.*

These facts apply to all logarithms, no matter what the base (see below). Moreover, in view of the fundamental concept of logarithms as exponents, one should keep in mind that

- *logarithms and the arguments of logarithms are dimensionless numbers.*

Thus, in physics problems, the argument (y in log y) often appears as a ratio or product in which there is a cancellation of units. The next section explains how to deal with logarithms of products and ratios.

The arithmetic of logarithms. (The rules outlined in this section apply equally well to logarithms using other bases.) You will recall from the discussion of scientific notation in Review 1, that powers of ten are multiplied by adding exponents. Thus, in view of the fact that logarithms are actually exponents, the logarithm of a product is

$$\log(xy) = \log x + \log y\ .$$

Likewise, since division of powers is done by subtracting exponents,

$$\log(x/y) = \log x - \log y\ .$$

When the argument of a logarithm is a product of n identical factors, we get this important rule for the logarithms of a power function

$$\log x^n = n \log x\ .$$

This rule is general; n need be neither an integer nor positive. For instance, the logarithm of a square root $\log \sqrt{a} = \log a^{1/2} = (1/2)\log a$.

No algebraic simplifications exist for the expressions (log a)(log b) or log a/log b. An illustration of how logarithms may be manipulated is the following example:

Sound intensity I can be stated in terms of sound energy flow per unit area, i.e., watts/cm^2. Since audible intensities vary over such a wide range, however, a logarithmically defined "decibel" unit is used which is based on intensity *relative* to a "threshold" intensity $I_o = 10^{-16}$ W/cm^2), as follows:

$$\text{decibels (dB)} = 10 \log_{10}(I/I_o)\ .$$

In going from a quiet office to a busy city street, the increase in decibels is about 30 dB. By what factor does the sound intensity (W/cm^2) increase?

DISCUSSION: $30 \text{ dB} = \text{dB(office)} - \text{dB(street)}$

$$= 10 \log(I_{street}/I_o) - 10 \log(I_{office}/I_o)$$

(Note that the label "dB" is dropped in the equation which follows. The decibel, like the radian, is a dimensionless unit and is sometimes omitted in a calculation, once the meaning of the numbers is made clear.)

Dividing by 10 on both sides of the equation, and combining logs:

$$3 = \log \frac{I_{street}/I_o}{I_{office}/I_o} = \log \frac{I_{street}}{I_{office}}$$

Taking powers of 10 on both sides yields

$$10^3 = 10^{\log [I(street)/I(office)]} = I_{street}/I_{office} .$$

In other words, the intensity is a factor of 1000 greater in the street.

OTHER BASES

Base 2. Just as $\log_{10}x$ can be thought of as the mathematical inverse of 10^x, $\log_2 x$ is the inverse of 2^x. In other words

$$\log_2 2^x = x \qquad \textit{and} \qquad 2^{\log_2 x} = x.$$

Base 2 mathematics is important for the study of probabilities, computers, and related subjects. While $\log_2$ is not mentioned often in introductory physics, working some exercises with base 2 is helpful for understanding logs and exponentials. Since $\log_2 x$, like $\log_{10} x$, can be thought of as an exponent, the arithmetic outlined for base 10 holds true for this base (and other bases), as well.

Natural logs. As with the exponential function, there are often mathematical simplifications to be acheived by using e = 2.718.... as the base of the logarithmic function. A number of topics in introductory physics are discussed in terms of the function $\log_e x$ function, called the "natural log" and usually written *ln x* . As with other bases, this function is inverse to the corresponding exponential function e^x, so that

$$\ln e^y = y \qquad \textit{and} \qquad e^{\ln y} = y .$$

Also, as has been already mentioned, the "arithmetic" of natural logs (such as taking the log of a product) is identical to that used with other bases. The "naturalness" of using base e in logarithmic expressions is analogous to the naturalness of using radians when describing oscillations: mathematical derivations are often free of arbitrary constants.

Converting among bases. Usually the conversion of a logarithmic expression to a corresponding expression using another base, if needed, will be worked out for you in an introductory physics text. However, doing so yourself is excellent practice in learning to manipulate logarithmic expressions. Here is one example:

Write an equation which relates log x to ln x, i.e., $\log_{10}x$ to $\log_e x$.

DISCUSSION: If y = ln x, then $x = e^y$. Thus

$$\log x = \log e^y = y \log e = (\ln x)(\log e).$$

Using an electronic calculator, we find log e = log 2.718 = 0.434, so that

$$\log x = 0.434 \ln x .$$

LOGARITHMIC PLOTS

Graphs which plot the logarithm of quantities, rather than the quantities themselves, have the virtue of being able to display values over a very wide range. Also they are widely used for the analysis of experimental data.

Semi-log plots. In this type of graph, the logarithm of the dependent variable is plotted versus the independent variable. Usually, semi-log plots are made on specially ruled graph paper on which the vertical divisions are ruled on a logarithmically varying scale; the divisions are marked by values of the dependent variable, rather than by values of its logarithm. Semi-log plots are especially useful in analyzing data which varies exponentially, for instance according to $y = A\,e^{kx}$. Taking the natural log of both sides of this expression we get

$$\ln y = kx + \ln A\ .$$

This is a linear expression in x; plotting ln y versus x yields a straight line with slope $k = (\Delta \ln y)/\Delta x$. The intercept is ln A. Consider the following example:

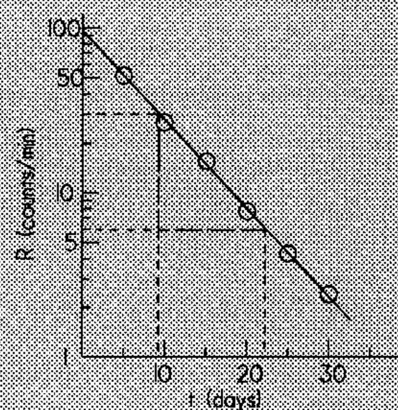
The table at the right gives the intensity of the radiation R (in "counts/minute") received by a radiation detector when it is placed near a sample of a radioactive substance. The radioactivity declines in intensity according to

$$R = R_o\,e^{-t/T},$$

t(days)	R(counts/min)
5	51
10	27
15	15.5
20	7.8
25	4.3
30	2.4

where R_o is the radiation intensity at $t = 0$ and T is called the "mean life" of the substance. Use a semi-log plot of the data to determine R_o and T.

DISCUSSION: Fit a straight line to the data points as shown. (Note that the vertical axis is scaled logarithmically and marked in terms of units of R, rather than log R.) The intercept gives R_o = 95 counts/min. In evaluating the slope, natural logs must be used to compute the "rise", since the exponential decay is expressed using base e. The slope is



$$-\frac{1}{T} = -\frac{\ln 30 - \ln 6.0}{(22 - 9)\text{days}} = -\frac{\ln 5.0}{13 \text{ days}}$$

Evaluating ln 5.0 and solving for T, we get mean life T = 8 days.

Log-log plot. Another logarithmic plot is used to analyze experimental data in which measurements of x are expected to depend on y according to some power law: $y = Ax^n$. Taking logarithms of both sides of this expression (any base can be used) gives

$$\log y = n \log x + \log A\ .$$

If a plot of log y versus log x yields a straight line, the power law is confirmed. The slope of this logarithmic plot $(\Delta \log y)/(\log x)$ is the power n. (Any logarithmic base is OK; problem 8 in the following drill illustrates this.)