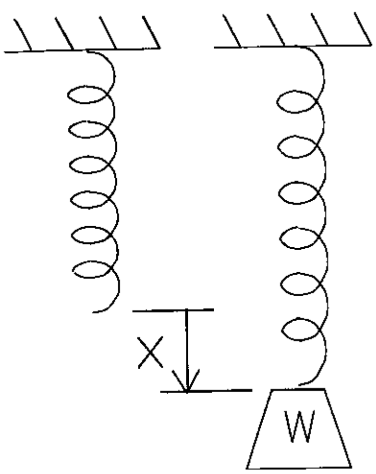
Review 4 — Proportion and Ratio

DIRECT PROPORTION

Definition and an example. Two quantities are said to be directly proportional to one another if an increase or a decrease of one of them by a certain factor is accompanied by an increase or decrease of the other one by the same factor. For example, if x is directly proportional to y, when x becomes twice as large, y also becomes twice as large. In symbols, direct proportionality between x and y can be written

$$y \propto x .$$

Physics abounds in phenomena for which measurable quantities are related in this way. A good example is the stretching of an ordinary helical spring, as pictured here. Suppose weights are to be suspended from such a spring. If the spring has a certain length to begin with, as weights are attached to the bottom of the spring the length will increase. Calling the amount of weight W (measured, for example, in pounds), and the corresponding increase in length x (measured, for example, in inches), for most springs it is found that W and x are directly proportional to one another.This is expressed



$$W \propto x .$$

Springs for which this relationship hold true are sometimes called "linear springs."

Proportionality constant. In every case of direct proportion, the relationship (as, for instance, between x and y) can also be described by an equation which looks like this:

$$y = kx,$$

where k is a quantity which depends neither on y nor x. k is called the "constant of proportionality".

Thus, the example given above of a spring and weight could have as well been described by the equation W = kx, where the proportionality constant k in this case is called the "spring constant." k is a quantitative measure of just how "stiff" or how "soft" the spring is.

A constant of proportionality must have units consistent with the other units in the proportionality equation. For instance, if the stretch of a spring is given in inches and the weight in pounds, the spring constant has units of pounds per inch (lbs/in). This is illustrated in the following example problem:

One end of a spring is held by a support and a fish is attached to the lower end. The spring constant is $5.0 \cdot 10^{-3}$N/m. What is the weight of the fish if the spring is stretched out an additional 3.0 cm from its normal length? (N stands for newtons, the S.I. unit of force; weight is merely gravitational force.)

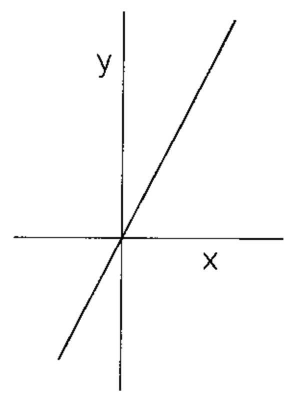
DISCUSSION: Assuming the spring to be "linear" we use the expression W = kx, where the proportionality constant k is the spring constant. In order to get W in appropriate units we must make a unit conversion (m to cm). Thus

$$\begin{aligned} W = kx &= (5.0 \cdot 10^{-3}\text{N/m})(3.0\ \text{cm}) \\ &= (5.0 \cdot 10^{-3}\text{N/m})(3.0 \cdot 10^{-2}\text{m}) \\ &= 1.5\ \text{N}\,. \end{aligned}$$

REPRESENTING DIRECT PROPORTION GRAPHICALLY

A straight line graph. Any equation relating quantities x and y corresponds to a curve drawn in a plane containing x and y axes (the "x-y plane").

When the relationship is one of direct proportion the curve is always a straight line passing through the origin.



Moving from point to point along this line a change of x by some factor (for instance a doubling of x) corresponds to a change of y by the same factor (a doubling of y).

Dependent and independent variables. The graph shown here illustrates a convention which is often used in plotting the equations of physics. The value of one of the variable quantities (the "dependent variable", in this case y) is thought of as being set by whatever values are assigned to the other variable quantities in the equation (the "independent variables", in this case just x). Usually the dependent variable is plotted along the vertical axis, while independent variables (one for each curve) are plotted along the horizontal axis.

Slope and proportionality constant. Changes in the value of an independent variable give rise to changes in the value of the dependent variable. An important fact about these changes is the following:

For a proportional relationship such as y = kx the changes in the variables are also proportional.

In the graph of the relationship y = kx, shown at the top of the next page, the change from one value of x to another has been labelled Δx, often called the "run." The corresponding change in y is labelled Δy, often called the "rise". It turns out that the constant of proportionality relating the rise and the run is also k. Thus we can write:

$$\Delta y = k\,\Delta x\,.$$

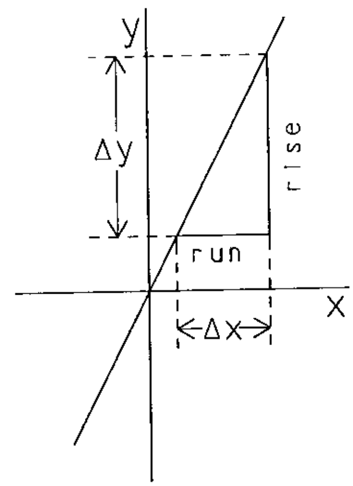
A measure of the steepness of the straight line is the ratio

$$\Delta y/\Delta x = k.$$

This ratio is called the "slope" of the straight line.

In the graph representing a proportional relationship the proportionality constant is equal to the slope of the line.

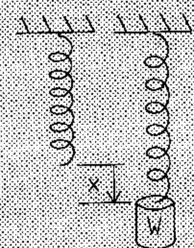
Slope may be negative as well as positive; in this Review we will consider only positive slopes.



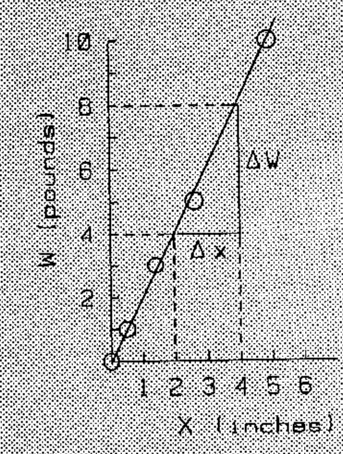
Again consider a linear spring, as in this example:

Below is a table giving data on the stretch x of a spring obtained when various weights W are suspended from it.

W(lb)	x(in)
0	0
1.0	0.50
3.0	1.49
5.0	2.51
10.0	4.99



Plot the points corresponding to each pair of values (W,x). Draw a straight line which fits the data and determine the spring constant by evaluating the slope of this line.



DISCUSSION: The deflection of spring is often used to determine force. Thus it is usual to consider x to be the independent variable and plot it on the horizontal axis. Set up appropriate scales on both axes and with a ruler draw a straight line which goes through the origin (0,0) and passes as close as possible to all the points. Then mark off with the ruler a convenient size run Δx (here 2.0 in has been chosen); next mark off the corresponding ΔW. The determination of k is as follows:

$$k = \frac{\text{rise}}{\text{run}} = \frac{\Delta W}{\Delta x} = \frac{4.0 \text{ lb}}{2.0 \text{ in}} = 2.0 \text{ lb/in}.$$

SOLVING PROBLEMS USING RATIOS

Setting up a ratio equation. Very often it isn't necessary to find the constant of proportionality in order to solve problems involving proportional relationships. When variable y changes, variable x changes by the same factor no matter what the proportionality constant may be. Thus if we know the ratio of a pair of independent variables we also know the ratio of the corresponding pair of dependent variables.

Labelling corresponding values of x and y by the same subscripts this relationship between ratios can be stated this way:

$$y_1/y_2 = \text{[the same factor]} = x_1/x_2 \,.$$

Sometimes you will see this written y_1:y_2 = x_1:x_2, but the ratio equation above is more useful.

For a proportional relationship the ratio of a pair of values of the dependent variable is *equal* *to the ratio of a corresponding pair of values of the independent variable.*

Using ratio equations. In many problems in which $y \propto x$, three of the four quantities x_1, x_2, y_1, and y_2 are known, and we are asked to find the remaining quantity. A simple algebraic manipulation enables us to write an equation in which the unknown quantity is written in terms of the known quantities. We start by writing down the ratio expression

$$\frac{x_1}{x_2} = \frac{y_1}{y_2}.$$

Suppose x_1 is the quantity to be determined. The desired expression is obtained by multiplying both sides of the ratio equation by x_2. Since the x_2's on the left hand side cancel we are left with

$$x_1 = x_2 \frac{y_1}{y_2} \,.$$

This way of rearranging an equation is also called "cross multiplication" since a factor (x_2) from one side of the equation is transferred diagonally across the equal sign. (See Review 6.) This manipulation is used in the following example:

Suppose we hang a 2.0 lb fish from the lower end of a linear spring causing it to deflect 2.5 cm. A larger fish causes a deflection of 3.0 cm. How much does the larger fish weigh?

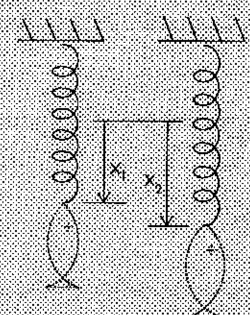
DISCUSSION: Using subscripted x's for the two deflections and subscripted W's for the corresponding weights, ratios can be set equal as follows:

$$x_2/x_1 = W_2/W_1 \,.$$

Thus

$$W_2 = W_1\,(x_2/x_1)$$

$$= (2.0\text{ lb})(3.0\text{ cm}/2.5\text{ cm}) = 2.4\text{ lb}\,.$$



(Using a calibrated scale to mark off the deflections of a spring directly in force units is the principle of the spring balance or "fish scale.")

Sometimes it is useful to start the solution of a ratio problem by writing

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} \,,$$

a form of the ratio equation which is easily obtained by applying cross multiplication to the expressions given above.