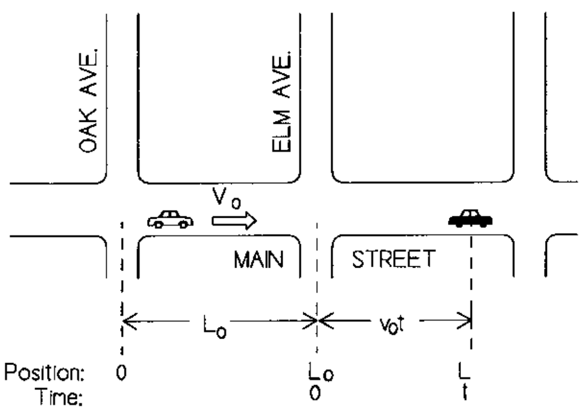
Review 5 — Linear Equations

THE GENERAL LINEAR EXPRESSION

Form of the equation. The equation describing the direct proportion is just a special case of the linear equation, one of the most useful algebraic relationships for describing physical phenomena. A general form of the linear equation is often written

$$y = mx + b$$

in which m and b are constants which do not depend on the variable quantities y and x. Whenever b is zero, this expression reduces to the equation of direct proportion, with proportionality constant m.



An application from physics. The motion of an object with constant speed along a straight line is easily described using a linear equation.

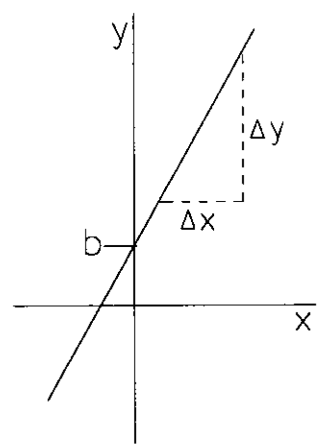
Imagine a car moving along Main Street with constant speed v_0. Let us use L to denote locations of the car as measured from the intersection of Main Street and Oak Avenue (where L=0). As the car passes the intersection with Elm Avenue (location L_0) a stopwatch to measure time t is turned on. All subsequent locations are given by simply adding to the location at t=0 (which is L_0) additional distances v_0t. Written as an equation we have

$$L = v_0t + L_0 .$$

In this equation L and t are the dependent and independent variables, respectively. v_0 corresponds to the constant m in the general expression given in the preceding paragraph; L_0 corresponds to b. Negative as well as positive values of t can be used in this equation; when negative values are inserted the calculated values of L are simply the distances of the car from Oak Avenue as it approached the intersection before the stopwatch was turned on.

GRAPHICAL INTERPRETATION

Graph of the general linear equation. In the x-y plane the graph of the linear function y = mx + b is a straight line, but unlike the direct proportion the line does not necessarily pass through the origin. This is illustrated in the figure at the right.



Interpreting the constants. As in the case of direct proportion changes in the independent variable (Δx) are proportional to the corresponding changes in the dependent variable (Δy). Instead of passing through the origin the line intersects the vertical axis at a value y = b, called the "y-intercept". The slope of line (the "rise"Δy divided by the "run" Δx) has the value m:

$$m = \text{slope} = \Delta y/\Delta x\ .$$

It is important to realize that both constants in the linear equation carry units; b has the same units as y, while the units of m must be a ratio of the units of y and x. In summary:

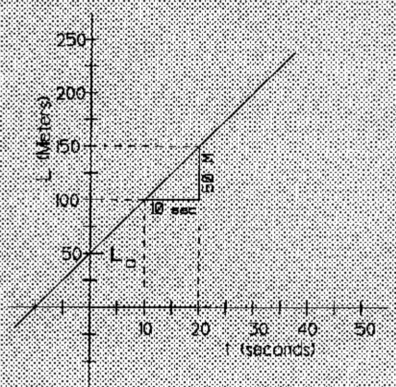
$y = mx + b$ plots as a straight line; m is the slope of the line and b is the y-intercept.

The following example applies these ideas to the case of the moving car described above:

(See the figure on the previous page.) Suppose that the intersection of Main Street with Elm Avenue is located 50 meters from the intersection with Oak Avenue. The car is travelling at a steady speed of 5.0 m/s.

(a) Write down the linear equation which describes the motion of the car (L vs. t) and draw the corresponding graph. (b) How far is the car from Oak Avenue when the stopwatch reads 30 s?

DISCUSSION: (a) The appropriate equation is $L = v_o t + L_o$



where $v_o = 5.0$ m/s and the position at t = 0 (Elm Ave.) $L_o = 50$ m. For the graph identify the slope with v_o, i.e., the rise must have 5.0 units for each 1.0 units of run. For convenience of scale the rise and run are plotted as 50 and 10 respectively, as shown at the left. Also the line must pass through the vertical axis at the intercept $L_o = 50$ m.

(b) For t = 30 s we have $L = v_o t + L_o$
$= (5.0\ \text{m/s})t + 50.\text{m}$
$= (5.0\ \text{m/s})(30\ \text{s}) + 50.\text{m}$
$= 200$ meters.

LINEAR DEPENDENCE WHERE THE VARIABLES CHANGE IN OPPOSITE DIRECTIONS

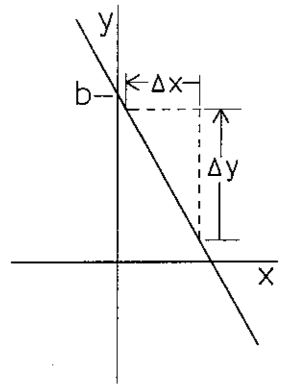
In all of the examples of direct proportion or of the general linear dependence given so far, increases in the independent variable (x) are accompanied by increases in the dependent variable (y). (Likewise in those examples decreases in x correspond to decreases in y.) However many situations occur in which *decreases* in one variable are proportional to *increases* in the other variable. For such cases the general form of the equations are just as given before:

$y = kx$ (direct proportion)
$y = mx + b$ (general linear dependence).

However,

- *when the variables change linearly in opposite directions the proportionality parameters k (or m) are* <u>*negative*</u> *rather than positive numbers.*

Negative slope. For negative k or m linear dependence is also represented by a straight line graph, but the straight line is tilted downwards along the horizontal axis rather than upwards. This is shown in the figure at the right. A positive (upward) "rise" of the line Δy corresponds to a negative (leftward) "run" Δx.



> *Linear (or proportional) dependence in opposite directions corresponds to a straight line graph with a negative slope.*

An example of constant speed motion can be used to illustrate linear but oppositely changing variables. This is the case in which the object moves towards lower values of the position variable as time goes on, i.e., when its motion is described by a negative speed, as in this situation:

Referring to the previous example, the car is returning at the same speed down Main Street moving towards Oak Avenue (v_o = -5.0 m/s). Again we time the motion so t = 0 when the car passes Elm Avenue (L_o = 50 m). When does the car reach Oak Avenue?

DISCUSSION: The question can be restated this way: find t when L = 0. Use the linear equation for the motion as follows:

$$L = v_o t + L_o$$

$$0 = (-5.0 \text{ m/s})t + 50 \text{ m}.$$

The value of t which satisfies this equation is t = 10 s.

Do not confuse this case with that of "inverse proportion", which is an entirely different relationship. Inverse proportion, described by y = k/x, will be taken up in Review 14.