Review 6 — Rearranging Equations - Basic Algebra

To find answers to problems in physics it is often necessary to do some algebraic manipulation. Most of the time the goal is to solve for an unknown, i.e., to isolate on one side of an equation the quantity which is to be determined, leaving the remaining quantities on the other side. Sometimes the purpose is to find an equivalent expression which is in a simpler or more familiar form. In this section the rules for rearranging algebraic expressions are reviewed and illustrated by applying them to linear equations.

MOVING QUANTITIES ACROSS THE EQUAL SIGN

Basic rule of equations. When an equation is rearranged, it is essential that the resulting expression yield the same solution. In other words the transformed equation must be "equivalent" to the original one. This will occur if the following rule is observed.

> *A mathematical operation which is applied to the one side of the equal sign must also be applied to the other side.*

Thus multiplying the entire left hand side ("lhs") by some number or expression is allowed as long as the same multiplication is carried out on the right hand side ("rhs")- likewise for the operations of division, addition, subtraction, raising to powers, or taking roots. This idea underlies the methods for moving quantities from one side of an equation to the other, outlined below.

Moving factors—cross multiplication. Sometimes we have a factor which multiplies or divides one *entire* side of an equation (not just a term on that side), and we want it to appear instead on the other side. To do this we can multiply or divide both sides by that factor in such a way that it cancels out on the original side. This operation was applied in Review 4 as a way of rearranging ratio expressions. As pointed out in that Review, it is convenient to regard this procedure as a "cross multiplication" which may be summarized this way:

> *Factors of one side of an equation can be moved diagonally across the equal sign, i.e., from the denominator on one side to the numerator on the other side, or vice versa.*

This is illustrated by the example below, where the factor (b+c) is changed from the denominator on the lhs to the numerator on the rhs:

$$\frac{a}{b+c} = \frac{d}{e} \quad \rightarrow \quad a = (b+c)\frac{d}{e}$$

As another example we have

$$\frac{(a)(b)}{c} = \frac{d}{e} \quad \rightarrow \quad a = \frac{(c)(d)}{(b)(e)}$$

Moving terms—transposition. We can also move *terms* from one side of an equation to the other. Suppose, for example, in the equation

$$a + b = c$$

we wish to move the term b to the rhs, leaving term a isolated on the lhs. This is done by subtracting b from both sides to get

$$a + b - b = c - b \qquad \textit{or} \qquad a = c - b.$$

This way of rearranging the equation can be conveniently thought of as a transposition of b from one side to the other with a change of sign, like this:

$$a + \overset{\curvearrowright}{b = c} - \qquad \textit{or} \qquad a = c - b.$$

The rule for transposition is simply stated:

Terms can be moved from one side of an equation to the other if accompanied by a change of sign.

ADDING, SUBTRACTING, MULTIPLYING, AND DIVIDING

Combining signs. Factors can be positive or negative. The following rule applies:

Multiplying or dividing numbers of like sign yields a positive result; otherwise the result is negative.

Thus, for example, the following three products all have the same value (+ab):

$$(+a)(+b) = (-a)(-b) = -(+a)(-b)\,.$$

Groups of quantities—associative rule. This paragraph and the following ones have to do with performing basic algebraic manipulations when there are groups of quantities in an expression, either a series of terms or strings of factors. The first rule has to do with how we can rearrange the parentheses which group the quantities together.

In a sum of three or more terms or a product of three or more factors it doesn't matter how we group the quantities within parentheses.

Subtraction and division are also covered by this rule since subtraction is simply addition of a negative number, and division is simply multiplication by the reciprocal (reciprocal $x \equiv 1/x$). The rule applied to addition and subtraction is illustrated by the following examples involving quantities a, b, and c:

$$a + b + c = (a + b) + c = a + (b + c)$$
$$a + b - c = a + (b - c).$$

A note of caution: In grouping terms within parentheses be careful about which terms are actually being subtracted. For instance a-b+c equals a+(-b+c); it does *not* equal a-(b+c).

The associative rule applied to multiplication and division is illustrated by these examples:

$$abc = (ab)c = a(bc)$$
$$(ab)/c = a(b/c)\,.$$

Groups of quantities—commutative rule.

Addition (and subtraction) or multiplication of a series of numbers doesn't depend on the order in which the operations take place.

This rule is also illustrated by combinations of a, b, and c:

$$a + b - c = a - c + b = b - c + a$$

and

$$abc = acb = bca\ .$$

Groups of quantities- distributive rule. This has to do with the way in which a complicated expression involving sums of terms inside parentheses (polynomials) can be reduced to a sum of relatively simple terms.

Multiplying (or dividing) a quantity consisting of a sum of terms inside a parenthesis by a number is the same as multiplying (or dividing) each of the terms by that number.

This is illustrated using quantities a, b, c, d as follows:

$$(a + b - c)d = ad + bd - cd$$
$$(a - b + c)/d = a/d - b/d + c/d\ .$$

Numerical examples. The above rules for manipulating groups of numbers may seem arbitrary and abstract, but actually they are simply statements about basic facts of arithmetic and the conventions for using parentheses. Any of these rules can be easily verified using numerical values, as in the following computations:

Association

$$2+7+10 = (2+7)+10 = 19 = 2+17 = 2+(7+10)$$
$$(3\times8)/6 = 24/6 = 4 = 3(4/3) = 3(8/6)$$

Commutation

$$12+7-5 = 19-5 = 14 = 7+7 = 12-5+7$$
$$8\times10\times2 = 80\times2 = 160 = 20\times8 = 10\times2\times8$$

Distribution

$$(20+1)2 = 21\times2 = 42 = 40+2 = 20\times2 + 1\times2$$
$$(11-4)/7 = 7/7 = 1 = 7/7 = 11/7 - 4/7$$

Multiplying polynomials by one another. A common algebraic situation is when two (or more) sums of terms within parentheses (polynomials) are multiplied together. The above principles of association, commutation, and distribution can be invoked (see exercise in Skill Drill 6) to show that this multiplication can be summarized in the following relatively simple rule:

Multiply each term of one polynomial by each term of the other polynomial, and add the products together.

Thus for example:

$$(a + b + c)(d + e) = ad + bd + cd + ae + be + ce .$$

Some particularly important special cases whose results are well worth remembering are the following:

$$(a \pm b)^2 = a^2 + b^2 \pm 2ab \quad \textit{and} \quad (a + b)(a - b) = a^2 - b^2 .$$

STEPS IN SOLVING FOR AN UNKNOWN—AN EXAMPLE DESCRIBING MOTION

In rearranging an equation to solve for an unknown, two things occur:

(1) All terms or factors containing the unknown are moved to the lhs, then
(2) all other terms or factors either cancel or are moved to the rhs.

The following problem shows how these two steps are typically carried out using the rules of algebraic manipulation which have been summarized in this Review.

The average speed v_{av} (v for "velocity") of a car moving from location L_1 to location L_2 is found by dividing the distance between the two positions by the time interval $(t_2 - t_1)$ elapsed during the motion. Expressed as an equation

$$v_{av} = \frac{L_2 - L_1}{t_2 - t_1}$$

Suppose the car is at $L_1 = 100$ m when a stopwatch reads 10 s. If the average speed is 10 m/s, what does the stopwatch read when the car is at $L_2 = 1000$ m ? In other words, what is t_2?

DISCUSSION: We are asked to solve for t_2, which requires that by algebraic manipulation we isolate it on the left hand side (lhs) of the expression.

The first step is to bring t_2 into the numerator on the lhs. This is done by *cross multiplying* with the factor $(t_2 - t_1)$. Then the *distributive rule* allows us to write

$$t_2 v_{av} - t_1 v_{av} = L_2 - L_1 .$$

After this all quantities (besides t_2) must be moved to the right hand side (rhs). Begin by *transposing* $(-t_1 v_{av})$ to the rhs yielding

$$t_2 v_{av} = L_2 - L_1 + t_1 v_{av} .$$

Next *cross multiply* to bring the v_{av} on the lhs to the denominator of the rhs. Finally dividing by that v_{av} according to the *distributive rule* gives us

$$t_2 = \frac{L_2 - L_1}{v_{av}} + t_1$$

The final quantitative answer is obtained by substituting the given numerical information: the car passes the 1000 m location when the stopwatch reads

$$t_2 = (1000 \text{ m} - 100 \text{ m})/(10 \text{ m/s}) + 10 \text{ s} = 100 \text{ seconds.}$$