Review 7 — Simultaneous Linear Equations

It would be convenient if every physics problem could be solved by finding an equation in which the unknown quantity was directly related to all the known quantities. Unfortunately in some cases more than one unknown quantity is involved in the algebraic statement of the problem. However, the problem often can still be solved using additional independent equations bearing on the physics. The main requirement is that there be at least as many of these "simultaneous equations" as there are unknown quantities. This Review discusses how this is done for situations in which all the equations are linear.

TWO SIMULTANEOUS LINEAR EQUATIONS

An example using a linear equation describing simple motion. An example of a linear equation given in Review 5 described the position L of an object moving in a straight path with constant speed v_o. It was written

$$L = v_o t + L_o$$

where L_o is the position when the time $t = 0$. Now consider a case in which two such equations are used to describe a situation, as follows:

Bob and Ray are each driving their pickups down the freeway in the same direction at constant speeds. Bob passes milepost 10 going 50 mi/hr at exactly 2:00 PM. At that same moment Ray passes milepost 8 going 60 mi/hr. At what milepost position does Ray overtake Bob? At what time does this occur?

DISCUSSION: We can write one equation by using information about the motion of Bob's car and a second one by using information about Ray's car. Choose 2:00 PM to be time $t = 0$. Using x to label position instead of L (a common convention), the motion of each of the two cars is described by these linear equations:

$$x_B = (50 \text{ mi/hr})t + 10.0 \text{ mi}$$

and

$$x_R = (60 \text{ mi/hr})t + 8.0 \text{ mi} .$$

According to the problem at a certain time t Bob and Ray are at the same place (call this simply x). Thus at that particular time and place the following two equations hold true:

$$x = (50 \text{ mi/hr})t + 10.0 \text{ mi}$$

and

$$x = (60 \text{ mi/hr})t + 8.0 \text{ mi} .$$

This is a pair of simultaneous linear equations in two unknowns (x and t) which can be solved to give the answers to the problem.

Basic strategy for finding the solutions. Described in broad terms, solving a pair of simultaneous linear equations in two unknowns proceeds as follows:

The two equations are reduced to a single equation in one unknown; the solution of this equation is then substituted back into one of the original equations to produce a second equation, which is then solved for the other unknown.

Two essentially equivalent procedures for carrying out this strategy are given below.

Solving by substitution of an equation. A straightforward approach which can be readily applied in most cases has these steps:

(1) Rearrange one equation so that one unknown is given in terms of the other (in the example above, for instance, t in terms of x); then

(2) use that expression to completely replace one of the unknowns in the other equation (for example, replace t).

The resulting equation contains only one unknown. Now carry out the remainder of the strategy which was summarized above, as follows:

(3) Solve the equation found in step (2) for the unknown (x).
(4) Substitute the value which was found for this unknown into one of the original equations, and
(5) solve this equation for the other unknown (t).

Here are how these steps are applied to the example above:

DISCUSSION: Refer to the two simultaneous equations in the previous example. *(Step 1)* Using the second equation to get t in terms of x we have

$$t = \frac{x - 8.0\text{mi}}{60\ \text{mi/hr}}$$

(Step 2) This is substituted into the first equation to get

$$x = \frac{(50\ \text{mi/hr})(x - 8.0\ \text{mi})}{60\ \text{mi/hr}} + 10\ \text{mi}$$

or

$$x = (5/6)x - (20/3)\ \text{mi} + 10\ \text{mi}\ .$$

(Step 3) The solution of this equation is $x = 20$ mi (mile post 20).

(Step 4) To find t substitute this value of x into the first of the simultaneous equations to get

$$20\ \text{mi} = (50\ \text{mi/hr})t + 10\ \text{mi}$$

which can be solved *(Step 5)* to give

$$t = 1/5\ \text{hr} = 12\ \text{minutes}.$$

Ray overtakes Bob at 2:12 PM.

Solving by addition or substraction of equations. An alternative method of solving the pair of simultaneous equations differs from the one outlined above only in its approach to eliminating one of the unknowns (Steps 1 and 2). This technique works nicely when the equations are relatively simple, and usually requires less algebraic manipulation than the substitution approach:

(1) Rearrange the two equations so that the dependence on one of the variables is contained in each of them in a single term of the same magnitude (negative or positive size).

(Usually this step is carried out by multiplying one of the equations by an appropriate factor which makes one of terms equal in size to the corresponding term in the other equation.) Next

(2) add or subtract the equations so as to eliminate the terms of equal magnitude.

The resulting equation contains just one unknown. The remainder of the procedure is just the same as outlined before (Steps 3-5). Here is this method applied to the previous example:

DISCUSSION: Referring to the pair of simultaneous equations in the example on p. 67 containing the two variables x and t, *(Step 1)* multiply the first equation by the fraction 6/5 so that the terms in t in both equations are of equal size. The two equations are now

$$(6/5)x = (60 \text{ mi/hr})t + 12 \text{ mi}$$

and

$$x = (60 \text{ mi/hr})t + 8.0 \text{ mi}\,.$$

Upon subtracting the second equation from the first *(Step 2)* the terms in t disappear leaving only one equation in x, viz.

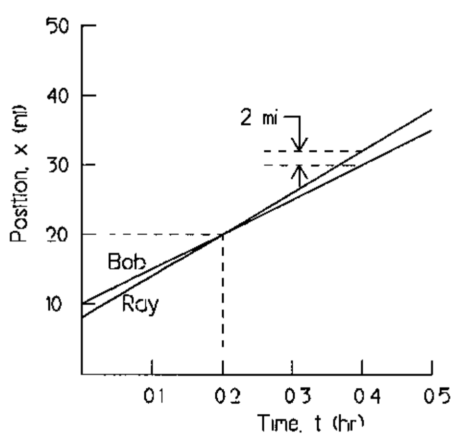
$$(1/5)x = 4.0 \text{ mi}.$$

Thus *(Step 3)*

$$x = 20 \text{ miles},$$

as before.

SIMULTANEOUS EQUATIONS REPRESENTED AS GRAPHS



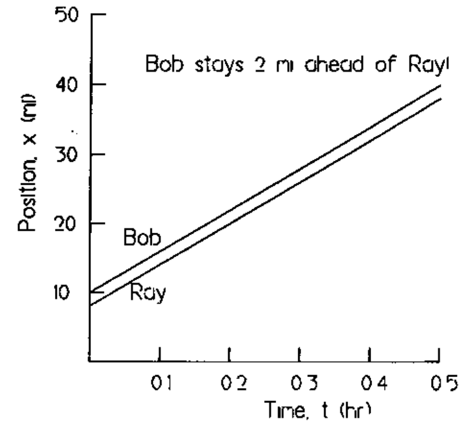
Graphs of equations can provide useful insight into the physics of a problem. A useful representation of a pair of simultaneous equations is a pair of curves plotted on the same graph. If there are just two variables the points where the curves cross each other correspond to the solutions of the equation pair.

Example of two linear equations. The example of the overtaking cars discussed above can be graphed as shown on the left. Each line represents one of the equations in the example. The equations are satisfied simultaneously for the pair of values where the lines cross (x = 20 mi, t = 12 min).

A graph like the one shown above can be used to answer other factual questions. For instance, if we were interested in when Ray will be 2 miles ahead of Bob, we can read this off directly by noting a value of t at which the lines are separated by 2 miles. (The answer is 24 minutes).

Possible solutions. The graphs representing two simultaneous equations also can provide insight into whether or not two solutions are to be expected.

If it turns out that the curves representing a pair of equations in two unknowns do not cross one another there can be no solution at all. For example, in the case of moving cars just discussed, if the two cars travel at the same speed the lines representing their motion would have the same slope; the lines would be parallel and never intersect. One car could never catch up with the other.



In contrast to the case of pairs of equations with no solutions are situations in which the corresponding curves cross at more than one point (more than one solution pair). This may happen when the equations are more complicated than linear. Nonlinear equations is a topic for a later Review.

Finally, it is worth realizing that if two curves fall directly on top of one another, the two equations they represent are, in fact, equivalent to one another. In such a case the two equations cannot be regarded as "independent" equations; they both contribute identically the same information to the solution of the problem. We would have to look for another relationship among the variables to find a solution.

MORE THAN TWO SIMULTANEOUS LINEAR EQUATIONS

Sometimes in setting up a problem you may need more than two simultaneous equations to represent the given information. In general, as long as there are as many independent equations as there are variables, a solution is possible. The procedure is, in essence, the same as that used to solve a pair of simultaneous equations: the set of equations is reduced to a single equation in one unknown, and the solution of that equation is then substituted back into the original equations. For example, a set of three linear independent equations in three unknowns can be dealt with as follows:

> *Use one of the three equations to eliminate one of the variables from the other two, by substituting for that variable wherever possible; then solve this pair of simultaneous equations as outlined above.*

In fact, the first example problem of this Review actually illustrates this procedure. In effect, two equations containing three variables (x_B, x_R, and t) were reduced, in the initial step, to two equations in two unknowns using a third equation $x_R = x_B = x$.